9/13 More on curve sketching
(12-1 & 12-2)

HW (due next Tuesday):

12-2 # 89, 91, 96

Vocabulary:
- diminishing returns: $\begin{cases} f' > 0 \\ f' \searrow \end{cases}$
- inflection point: f' switches between $\nearrow$ & $\searrow$
  (same as f switching ~~fro~~ between C.U. & C.D. at a point)

$f(x) = x^4 - 2x^2 + 7$

$f'(x) = 4x^3 - 2(2x) + 0 = 4x^3 - 4x$

$f''(x) = 4(3x^2) - 2(2)(1) = 12x^2 - 4$

Solve $0 = 4x^3 - 4x = 4x\underbrace{(x^2-1)}_{(x+1)(x-1)}$

$a^2 - b^2 = (a+b)(a-b)$

$ax^2 + bx + c = a(x - r_+)(x - r_-)$

$r_\pm = \dfrac{-b \pm \sqrt{b^2 - 4ac}}{2a}$

$$0 = 4x(x+1)(x-1) = f'(x)$$

$$\iff 4x = 0 \quad \text{or} \quad x+1 = 0 \quad \text{or} \quad x-1 = 0$$

$$x = 0 \quad \text{or} \quad x = -1 \quad \text{or} \quad x = 1$$

x	Test point -2	-1	Test point $-\frac{1}{2}$	0	Test point $\frac{1}{2}$	1	test point 2
$f'(x)$	-24	0	$+3/2$	0	$-3/2$	0	24
$f(x)$	$\searrow$	local min	$\nearrow$	local max	$\searrow$	local min	$\nearrow$

$$f'(-2) = 4(-2)\underbrace{(-2+1)}_{-1}\underbrace{(-2-1)}_{-3} = -24$$

$$f'(-\tfrac{1}{2}) = \underbrace{4(-\tfrac{1}{2})}_{-2}\underbrace{(-\tfrac{1}{2}+1)}_{\frac{1}{2}}\underbrace{(-\tfrac{1}{2}-1)}_{-\frac{3}{2}} = \frac{3}{2}$$

$$f'(\tfrac{1}{2}) = \underbrace{4(\tfrac{1}{2})}_{2}\underbrace{(\tfrac{1}{2}+1)}_{3/2}\underbrace{(\tfrac{1}{2}-1)}_{-1/2} = -3/2$$

$$f'(2) = 4(2)\underbrace{(2+1)}_{3}\underbrace{(2-1)}_{1} = 24$$

$$f(x) = x^4 - 2x^2 + 7$$

$$f(-1) = \underbrace{(-1)^4}_{1} - 2\underbrace{(-1)^2}_{1} + 7 = 6 = \text{local min.}$$

$$f(0) = 0^4 - 2(0)^2 + 7 = 7 = \text{local max.}$$

$$f(1) = 1^4 - 2(1)^2 + 7 = 6 = \text{local min.}$$

Concavity & inflection points

$f''(x) = 12x^2 - 4$ Solve

$$0 = 12x^2 - 4$$
$$4 = 12x^2$$
$$\frac{1}{3} = x^2$$
$$\pm\sqrt{1/3} = x$$

x	Test point -1	$-\sqrt{\frac{1}{3}}$	Test point 0	$+\sqrt{\frac{1}{3}}$	Test point 1
$f''(x)$	$+8$	0	-4	0	$+8$
$f(x)$	C.U.	inflection point	C.D.	inflection point	C.U.

$$f''(-1) = 12\underbrace{(-1)^2} - 4 = 8$$
$$f''(0) = 12(0)^2 - 4 = -4$$
$$f''(1) = 12(1)^2 - 4 = 8$$

To get coordinates of inflection points, plug into $f(x)$

$$f\left(-\sqrt{\tfrac{1}{3}}\right) = \left(-\sqrt{\tfrac{1}{3}}\right)^4 - 2\left(-\sqrt{1/3}\right)^2 + 7$$
$$= \left(\left(-\sqrt{1/3}\right)^2\right)^2 - 2\left(-\sqrt{1/3}\right)^2 + 7$$
$$= \underbrace{(1/3)^2}_{1/9} - \underbrace{2(1/3)}_{6/9} + \underbrace{7}_{63/9} = \frac{58}{9}$$

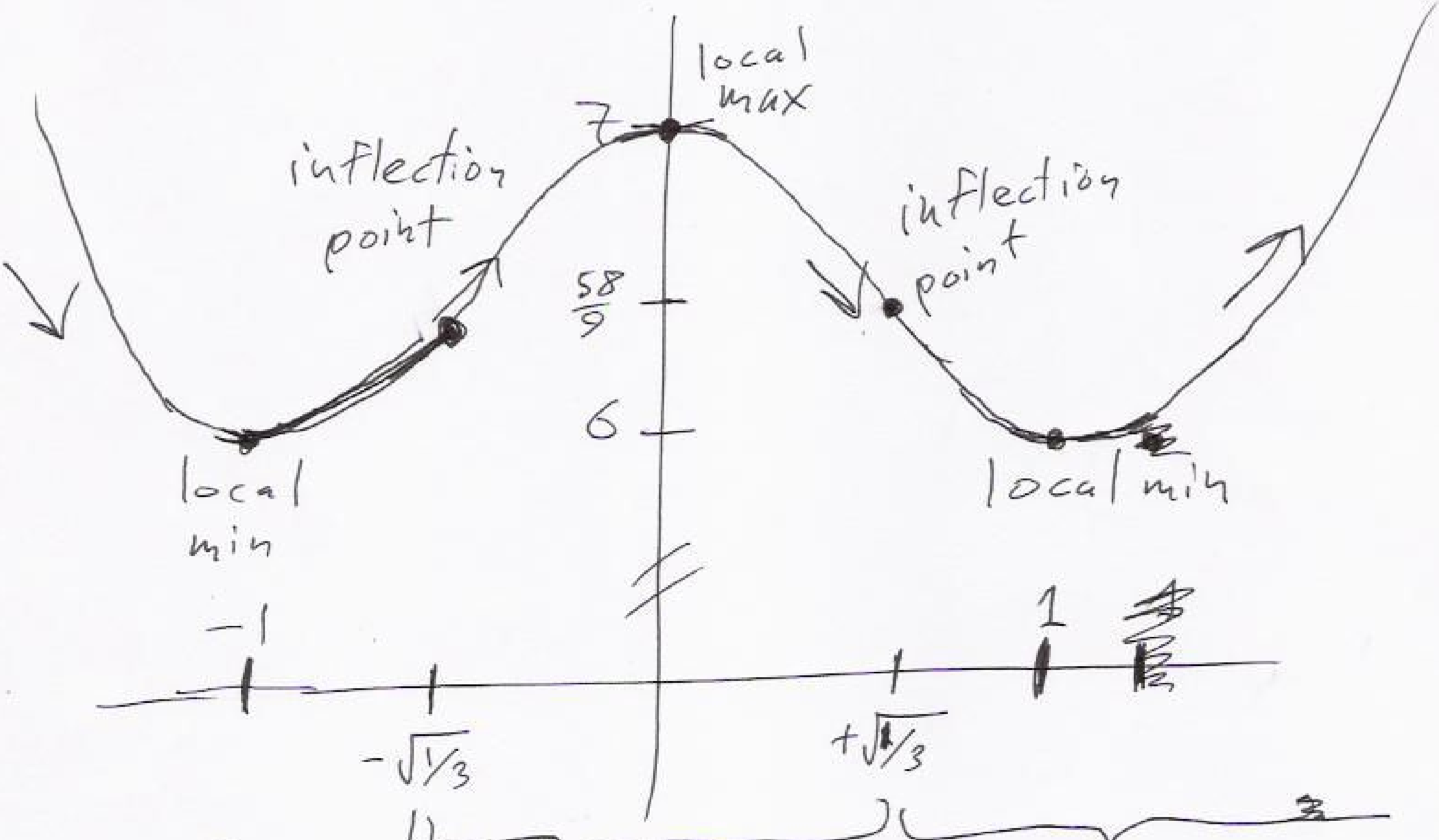
$$f(\sqrt{1/3}) = \left((+\sqrt{1/3})^2\right)^2 - 2(\sqrt{1/3})^2 + 7$$
$$= (1/3)^2 - 2(1/3) + 7 = \frac{58}{9}$$

Inflection points $\left(\pm\sqrt{1/3}, \frac{58}{9}\right)$

points of local minima: $(\pm 1, 6)$

point of local maximum $(0, 7)$

$$\pm\sqrt{\tfrac{1}{3}} \approx \pm 0.577\ldots \qquad \frac{58}{9} = \frac{54}{9} + \frac{4}{9} = 6.444\ldots$$



C.U. on $(-\infty, -\sqrt{1/3}]$

C.D. on $[-\sqrt{1/3}, \sqrt{1/3}]$

C.U. on $[\sqrt{1/3}, \infty)$

f is $\nearrow$ on ~~$[-\sqrt{1/3}, 0]$ &~~

$[-1, 0]$ & $[1, \infty)$

f is $\searrow$ on $(-\infty, -1]$ & $[0, 1]$

Exam: bring calculator & 1 sheet of notes (double-sided)

(Problems similar to H/W.)

Topics from class days 8/25 — 9/13.

Global min & max of $f(x)$?

Look at $f(+\text{big})$ & $f(-\text{big})$

$$f(x) = x^4 - 2x^2 + 7$$

$$f(+\text{big}) = \underbrace{(+\text{big})^4}_{+\text{big}} \underbrace{- 2\underbrace{(+\text{big})^2}_{+\text{big}}}_{-\text{big}} + 7$$

which is bigger?

$$f(x) = x^4\left(\frac{x^4}{x^4} - \frac{2x^2}{x^4} + \frac{7}{x^4}\right) = x^4\left(1 - \frac{2}{x^2} + \frac{7}{x^4}\right)$$

$$f(+\text{big}) = (+\text{big})^4\left(1 - \frac{2}{(+\text{big})^2} + \frac{7}{(+\text{big})^4}\right)$$

$$= (+\text{big})\left(1 - \frac{2}{+\text{big}} + \frac{7}{+\text{big}}\right)$$

$$= (+\text{big})\underbrace{\left(1 - \text{small} + \text{small}\right)}_{\approx 1}$$

$$\approx (+\text{big})$$

$f(+\text{big}) = +\text{big} \Rightarrow$ no global max.

Similarly, $f(-\text{big}) = +\text{big}$.

Since f gets large & positive when $x = \pm\text{big}$ (i.e., x is large), there is no global maximum, but 6 is the global minimum.

$f(x) = x + \frac{1}{x} = x + x^{-1}$

$f'(x) = (x + x^{-1})' = 1 + (-1)x^{-2}$

$f'(x) = 1 - \frac{1}{x^2}$ not defined at 0.

Solve $0 = 1 - \frac{1}{x^2}$

$x^2 \cdot 0 = x^2\left(1 - \frac{1}{x^2}\right)$

$0 = x^2 - 1$

$1 = x^2$

$f'(x) = 1 - \frac{1}{x^2}$ $\quad \pm 1 = x \quad$ $f(x) = x + \frac{1}{x}$

x	-2	-1	$-\frac{1}{2}$	0	$\frac{1}{2}$	1	2
$f'(x)$	$3/4$	0	-3	undef	-3	0	$3/4$
$f(x)$	$\nearrow$	-2	$\searrow$	undef	$\searrow$	2	$\nearrow$
		local max				local min	

Then look at $f''(x)$ & concavity...