Exponentials & derivatives (11-1)
(11-2)

Δx = change in x

Δx = absolute change in x

$\frac{\Delta x}{x}$ = relative change in x

old $x = 125$

new $x + \Delta x = 150$

$\Delta x = 150 - 125 = 25$

$(\Delta x)/x = 25/125 = 0.2 = 20\%$

Suppose a stock doubles in price in a year. old x new $x + \Delta x = 2x$

$\Delta x = 2x - x = x$ $(\Delta x)/x = x/x = 1 = 100\%$

Δt = change in time = 1 (years)

t	x	$(\Delta x)/x$
0	100	10%
0.1	110	10%
0.2	121	10%
0.3	133.1	10%
0.4	146.41	10%
0.5	[illegible]	10%
0.6	⋮	10%
0.7	⋮	10%
0.8	⋮	10%
0.9	⋮	10%
1.0	259.37	

$$\overbrace{100 \cdot (110\%)^{10}}^{100 \cdot 1.1^{10}}$$

If $\Delta t = 1/100$, and still $\Delta x = x\,\Delta t$, then, starting at $x = 100$ at $t = 0$, x grows by $\frac{1}{100} = 1\%$ every time interval of $\Delta t = \frac{1}{100}$. When $t = 1$, $x = \underbrace{100\,(1.01)^{100}}_{\approx 270.48}$,

$$\frac{\Delta x}{x} = 1 \text{ \& } \Delta t = 1 \Rightarrow \frac{\Delta x/\Delta t}{x} = 1$$

$$\Rightarrow \frac{\Delta x}{\Delta t} = x \Rightarrow \boxed{\Delta x = x \Delta t}$$

Now consider, instead of 100% growth over 1 year, 50% growth over six months, twice.

$$\Delta t = 1/2 = 1/2 \text{ year} = 6 \text{ months}$$

$$\Delta x/x = 50\% = 1/2 = \Delta t$$

$$\Delta x/x = \Delta t \Rightarrow \boxed{\Delta x = x\Delta t}$$

t	x	$(\Delta x)/x$		t	x	$(\Delta x)/x$
0 (start)	100			0	100	
						50%
		100%	vs	$1/2$	150	
						50%
1 (end)	200			1	225	

Both satisfy $\Delta x = x\Delta t$

t	x
0	100
⋮	⋮
1	271.69

$\Delta t = 1/1000$

$\Delta x = x \Delta t$

$100(1.001)^{1000}$

$$\begin{cases} 100(1.0001)^{10000} = 271.81\ldots \\ \Delta t = 1/10000 \ \& \ \boxed{\Delta x = x \Delta t} \end{cases}$$

As Δt gets smaller & smaller, our sequence 200, 225, 259.37, 270.48, 271.69, 271.81, ... tends to $100e$.

Fancy notation: $\lim\limits_{\Delta t \to 0} 100(1+\Delta t)^{1/\Delta t} = 100e$

$dx = x\,dt$ describes this limiting process

If $\boxed{dx = x\,dt}$ then x at time $t=1$ is e times x at time $t=0$.

$x(1) = e\,x(0)$.

More generally, $x(t) = e^t x(0)$

Special case: $x(0) = 1 \Rightarrow x(t) = e^t$

$\Rightarrow d(e^t) = e^t\,dt$

$\Rightarrow (e^t)' = \dfrac{d(e^t)}{dt} = \dfrac{e^t\,dt}{dt} = e^t$

If the US population is growing at a rate of 1% per year and we model this as continuous growth, then give a population of 300,000,000 now, what will be the population in 2 years?

x = population $\quad t$ = time in years

$t=0$ now $\quad x(0) = 300{,}000{,}000$

When $\Delta t = 1$, $(\Delta x)/x = 1\% = 1/100$ is the "discrete" model:

$$\Delta x = \frac{1}{100}x = \frac{x \cdot 1}{100} = \frac{x \, \Delta t}{100}.$$

~~Discr~~ Continuous model: $dx = \frac{x \, dt}{100}$

$$u = t/100 \Rightarrow dx = x \, du$$

$$\Rightarrow x = x(0) e^{u} = x(0) e^{t/100}$$

when $t = 2$, $x = 300{,}000{,}000 \, e^{2/100}$

$$x = 30{,}606{,}040$$

$$\boxed{\begin{array}{ll} \text{Shortcut:} & dx = kx \, dt \\ \text{implies} & x = x(0) e^{kt} \end{array}}$$

If something continuously grows at a rate of 10% per year, how long does it take for it to double? x = something t = time

What is t when $x(t) = 2x(0)$?

$dx = \frac{1}{10} x \, dt \Rightarrow x = x(0) e^{(\frac{1}{10} t)}$.

Solve $2x(0) = x(0) e^{(t/10)}$

$2 = e^{t/10}$ $\qquad \ln(e^y) = y$

$\ln 2 = t/10$ $\qquad e^{\ln y} = y$

$10 \ln 2 = t$

6.93 ... years is the doubling time.

What is $(\ln x)'$?

$y = \ln x \Rightarrow e^y = x \Rightarrow d(e^y) = dx$

$\Rightarrow e^y \, dy = dx \Rightarrow x \, dy = dx$

$\Rightarrow dy = dx/x \Rightarrow d(\ln x) = dx/x$

$\Rightarrow (\ln x)' = \frac{d(\ln x)}{dx} = \frac{dx/x}{dx} = 1/x$

$d(e^x) = e^x dx$	$(e^x)' = e^x$
$d(\ln x) = dx/x$	$(\ln x)' = 1/x$

$(5e^{3t})' = ?$

$d(5e^{3t}) = 5\,d(e^{3t}) = 5e^{3t}\,d(3t)$

$d(5e^{3t}) = 15e^{3t}\,dt$

$(5e^{3t})' = \dfrac{d(5e^{3t})}{dt} = 15e^{3t}$

More generally, if k is a constant,

$(e^{kx})' = ke^{kx}$

$(e^{x^2})' = ?$

$d(e^{x^2}) = e^{x^2}\,d(x^2) = e^{x^2}(2x\,dx)$

$(e^{x^2})' = \dfrac{d(e^{x^2})}{dx} = e^{x^2}\,2x$

More generally, $(e^{f(x)})' = e^{f(x)} f'(x)$.

HW: #1 Carbon-14 in a buried fossil decays radioactively. The relative rate of change of the amount of ^{14}C is -50% per 5700 years

Find k such that if t is measured in years, and x is the amount of carbon-14, then

$$x(t) = x(0)e^{-kt} \text{ satisfies}$$

$$x(5700) = x(0)/2.$$

HW #2

$$(xe^{3x})' = ?$$

$$(x^2 \ln(5x))' = ?$$

$$(e^x/\ln x)' = ?$$