# Logarthmic differentiation & Elasticity (11-7)

small relative change in $x \approx dx/x = d(\ln x)$
small (absolute) change in $x \approx dx$
small (absolute) change in $\ln x \approx d(\ln x) = dx/x$

Recall: $\ln(ab) = \ln a + \ln b$
$\ln(a/b) = \ln a - \ln b$
$\ln(a^b) = b \ln a$

New: $d(\ln x) = dx/x$

$$\left( \frac{(x+3)^5 \sqrt{1-x^2}}{(e^x - x)^3} \right)' = ?$$

$$y = \frac{(x+3)^5 \sqrt{1-x^2}}{(e^x - x)^3} \qquad y' = \frac{dy}{dx}$$

Logarthmic differentiation: find $\frac{dy}{y} = d(\ln y)$ and use that to find $y'$.

$$\ln y = \ln (x+3)^5 + \ln (1-x^2)^{1/2} - \ln (e^x - x)^3$$

$$\ln y = 5 \ln (x+3) + \frac{1}{2} \ln (1-x^2) - 3 \ln (e^x - x)$$

$$\frac{dy}{y} = d(\ln y) = 5\, d(\ln(x+3)) + \frac{1}{2}\, d(\ln(1-x^2)) - 3\, d(\ln(e^x - x))$$

$$\frac{dy}{y} = 5 \frac{d(x+3)}{x+3} + \frac{1}{2} \frac{d(1-x^2)}{1-x^2} - 3 \frac{d(e^x - x)}{e^x - x}$$

$$\frac{dy}{y} = \underbrace{5 \frac{dx + 0}{x+3} + \frac{1}{2} \frac{0 - 2x\,dx}{1-x^2} - 3 \frac{e^x dx - dx}{e^x - x}}$$

$$y' = \frac{dy}{dx} = \boxed{\frac{dy}{y}} \cdot \frac{y}{dx} \qquad y = \frac{(x+3)^5 \sqrt{1-x^2}}{(e^x - x)^3}$$

$$y' = \left(5 \frac{1}{x+3} + \frac{1}{2} \frac{-2x}{1-x^2} - 3 \frac{e^x - 1}{e^x - x}\right) \frac{(x+3)^5 \sqrt{1-x^2}}{(e^x - x)^3}$$

| products | $\xrightarrow{\ln}$ | sums |
|---|---|---|
| quotients | | differences |
| constant powers | | constant multiples |

HW Find the derivatives of the following functions:

① $f(x) = (x^3 - x + 1)^5 (\ln x + x^2)^3 x^2$

② $g(x) = (e^{3x} - 2)(e^{5x^2} + x)\sqrt{x+1}$

③ $h(x) = \dfrac{(\sqrt{x} + \sqrt[3]{x})\sqrt[3]{1+x^3}}{(5x+3x^2)^2 (x-2)}$

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Elasticities: ratios of relative changes. (11-7)

The X-elasticity of Y is

$$\frac{dY/Y}{dX/X} = \frac{d(\ln Y)}{d(\ln X)}.$$

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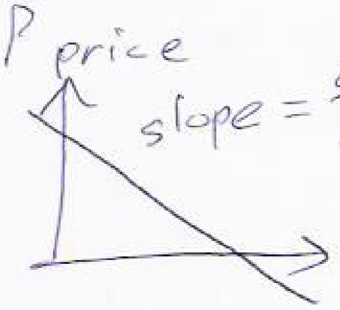
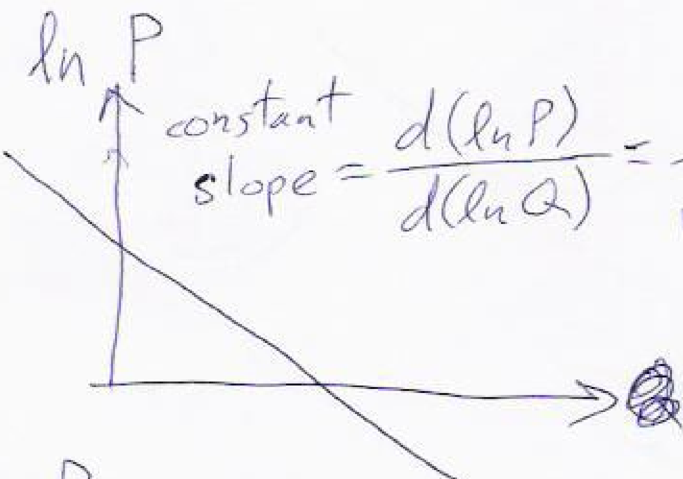
If your monthly gasoline consumption is 50 gallons when the price is \$3.50/gallon and 45 gallons when the price is \$4.00/gallon, then what would a linear demand model predict your monthly consumption be if the price were \$4.40?

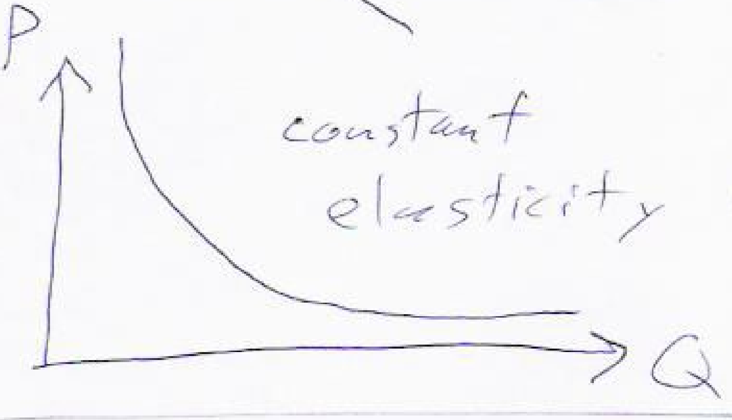



slope $= \frac{dP}{dQ} =$ constant

What would a constant elasticity demand model predict?




constant slope $= \frac{d(\ln P)}{d(\ln Q)} = \frac{1}{\text{price-elasticity}} \qquad \frac{dQ/Q}{dP/P}$ price-elasticity




constant elasticity demand model

Linear demand model

| P | Q |
|---|---|
| 3.50 | 50 |
| 4.00 | 45 |
| 4.40 | ? |

$$\frac{dQ}{dP} = \text{constant} = \frac{\Delta Q}{\Delta P} = \frac{50-45}{(4.00-3.50)}$$

$$= \frac{5 \text{ gallons/month}}{-\$0.50/\text{gallon}} = -10 \left(\frac{\text{dollar}\cdot\text{gal}^2}{\text{month}}\right)$$

$$\frac{\Delta Q}{\Delta P} = 10 \Rightarrow \Delta Q = 10\,\Delta P$$

Let $\Delta P$ be the change from 4.00 to 4.40.

$\Delta P = 0.40$. $\Delta Q = -10 \Delta P = -4$

Q changes from 45 to $45 - 4 = 41$

Elasticity-constant demand model:

$$\frac{d(\ln Q)}{d(\ln P)} = \text{constant} = \frac{\Delta(\ln Q)}{\Delta(\ln P)} = \frac{3.807 - 3.912}{1.386 - 1.253}$$

| P | ln P | Q | ln Q |
|---|---|---|---|
| 3.50 | 1.253 | 50 | 3.912 |
| 4.00 | 1.386 | 45 | 3.807 |
| 4.40 | 1.504 | ? | ? |

$$\frac{d(\ln Q)}{d(\ln P)} = -0.789$$

Let $\Delta(\ln P)$ be the change from 1.386 to 1.504

$$\Delta(\ln Q) = -0.789 \underbrace{\Delta(\ln P)}_{0.118} = -0.0929$$

ln Q changes from 3.807 to $\underbrace{3.807 - .0929}_{3.714}$

Q changes from 45 to $e^{3.714} = 41.006$

HW. Redo this problem with \$4.40 replaced by \$6, \$8, then \$10.

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U.S. population is currently 312,000,000 and growing at 3,000000 per year.

U.S. vehicles miles driven (per month) is currently 259 billion and decreasing at 3.8 billion per year.

What is the rate of change of vehicle miles driven per month per person?

$M = \cancel{3.8\times10^9}\ 259\times10^9 \qquad \frac{dM}{dt} = -3.8\times10^9$

$t$ = time in years

$P = 312\times10^6 \qquad \frac{dP}{dt} = 3\times10^6$

$\frac{d(M/P)}{dt} = ?$ Quotient rule:

$$d(M/P) = \frac{dM\cdot P - M\,dP}{P^2}$$

$$\frac{d(M/P)}{dt} = \frac{\frac{dM}{dt}\cdot P - M\frac{dP}{dt}}{P^2} = \underline{-2.02}$$

Logarithmic way:

$$\frac{d(M/P)}{dt} = ? \quad \text{Start with } d(\ln(M/P))$$

$$d(\ln(M/P)) = d(\ln M - \ln P)$$

$$= d(\ln M) - d(\ln P) = \frac{dM}{M} - \frac{dP}{P}$$

$$\frac{d(\ln(M/P))}{dt} = \underbrace{\frac{dM/dt}{M}}_{-1.47\%} - \underbrace{\frac{dP/dt}{P}}_{0.962\%} = -2.43$$

$$\frac{d(M/P)/(M/P)}{dt} = \frac{d(M/P)/dt}{M/P}$$

relative rate of change

$$\frac{d(M/P)}{dt} = (M/P)(-2.43\%) = -20.2$$