Optimization ← 12-6

derivative rules (including implicit diff.) ← 11-1, 11-2, 11-4 (e^x, $\ln x$), 11-4 ← chain rule, implicit diff. ← 11-5

elasticity ← 11-7

related rates ← 11-6

---

n constant $\Rightarrow$ $d(x^n) = nx^{n-1}dx$

$(x^n)' = nx^{n-1}$

$(x^2)' = 2x \quad | \quad d(x^2) = 2x\,dx$

$(x)' = 1 \quad | \quad dx = 1\,dx$

$d(f^n) = nf^{n-1}df$

$(f(x)^n)' = nf(x)^{n-1}f'(x)$

$df = f'(x)\,dx$

$d((x^2+1)^3) = 3(x^2+1)^2\underbrace{d(x^2+1)}_{2x\,dx}$

$d(f^3) = 3f^2df$

$((x^2+1)^3)' = 3(x^2+1)^2(2x)$

$(x^{1/2})' = \frac{1}{2}x^{-1/2}$

$\sqrt{x}\,' = \frac{1}{2\sqrt{x}}$

$d(\sqrt{x}) = \frac{dx}{2\sqrt{x}}$

$(x^{-1})' = (-1)x^{-2}$

$\left(\frac{1}{x}\right)' = -\frac{1}{x^2}$

$d\left(\frac{1}{x}\right) = \frac{-dx}{x^2}$

$d(x^3) = 3x^2dx$

$(x^3)' = 3x^2$

Other rules: $d(fg) = df \cdot g + f \cdot dg$

$$d\left(\frac{f}{g}\right) = \frac{df \cdot g - f \cdot dg}{g^2}$$

k constant $\Rightarrow$ $d(kf) = k\,df$

~~constant~~

$$d(f/k) = df/k$$

$$d(\ln f) = df/f$$

$$d(e^f) = e^f\,df$$

$$d(\ln(x^3 - x)) = \frac{d(x^3 - x)}{x^3 - x}$$

$$= \frac{3x^2dx - dx}{x^3 - x}$$

$$(\ln(x^3 - x))' = \frac{3x^2 - 1}{x^3 - x}$$

$$d(e^{-x^2}) = e^{-x^2}\,d(-x^2)$$

$$= e^{-x^2}(-2x\,dx)$$

$$(e^{-x^2})' = -2xe^{-x^2}$$

$$d(f + g) = df + dg$$

$$d(f - g) = df - dg$$

$$\underbrace{d(f(x))}_{\text{differential}} = \underbrace{f'(x)}_{\text{derivative}}\, dx$$

$$d(f(g(x))) = f'(g(x))\, d(g(x))$$
$$= f'(g(x))\, g'(x)\, dx$$

$$\boxed{(f(g(x)))' = f'(g(x))\, g'(x)}$$

$$(f \circ g)' = (f' \circ g) \cdot g'$$

Chain rule for derivatives

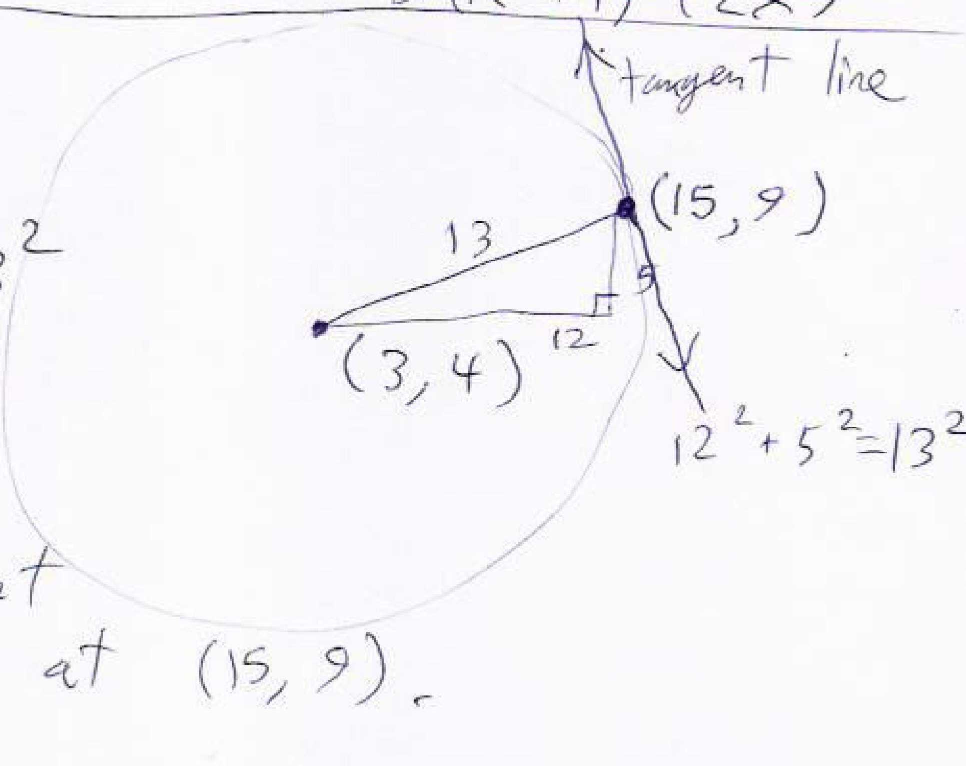
$$((x^2+1)^3)' = 3(x^2+1)^2 (x^2+1)' =$$

$(x^3)' = 3x^2$

$$3(x^2+1)^2(2x)$$

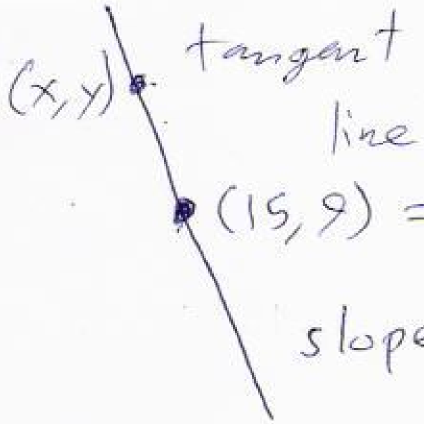
Circle

$$(x-3)^2 + (y-4)^2 = 13^2$$



$12^2 + 5^2 = 13^2$

Find the equation for the line tangent to the circle at (15, 9).



$(15,9) = (x_0, y_0)$

slope $m = ?$

$$\frac{\Delta y}{\Delta x} = m; \quad \frac{\Delta y}{\Delta x} = \frac{y-y_0}{x-x_0}$$

$$m = \frac{y-9}{x-15}$$

$$y-9 = m(x-15)$$

$m = \frac{dy}{dx}$ where $(x,y) = (15,9)$

$(x-3)^2 + (y-4)^2 = 13^2$ (all of the circle)

$$d\left[(x-3)^2 + (y-4)^2\right] = \underbrace{d(13^2)}_{0}$$

$$d(f^2) = 2f\,df$$

$$2(x-3)\underbrace{d(x-3)}_{dx - \underbrace{d3}_{0}} + 2(y-4)\underbrace{d(y-4)}_{dy - \underbrace{d4}_{0}} = 0$$

$$2\left((x-3)dx + (y-4)dy\right) = 0$$

$$(x-3)dx + (y-4)dy = 0$$

$$(x-3) + (y-4)\frac{dy}{dx} = 0$$

$$(15-3) + (9-4)\frac{dy}{dx} = 0$$

$$12 + 5\frac{dy}{dx} = 0$$

$$m = \frac{dy}{dx} = -\frac{12}{5} = -2.4$$

Tangent line: $y-9=-2.4(x-15)$

Elasticity: The X-elasticity of Y is $\dfrac{dY/Y}{dX/X}$, which equals $\dfrac{d(\ln Y)}{d(\ln X)}$.

If a product, say, a car, has income elasticity 0.13 and current demand for the car is 18,000 (cars) and current income is \$40,000. If income decreases to \$37,000, estimate the new demand for the car.

Use $\dfrac{d(\ln D)}{d(\ln I)}$ to estimate $\dfrac{\Delta(\ln D)}{\Delta(\ln I)}$

D = demand; I = income

D (old)	ln D (old)	$\Delta(\ln D)$	$\ln D + \Delta(\ln D)$
18000	9.798127	?	?

$$\frac{d(\ln D)}{d(\ln I)} = 0.13 \Rightarrow d(\ln D) = 0.13\, d(\ln I)$$

$$\Delta(\ln D) \approx 0.13\, \Delta(\ln I)$$

I (old) : 40000 $\quad \ln I = \ln(40000) = 10.596634$

(new) : 37000 $\quad \ln I + \Delta(\ln I) = \ln(37000) \Rightarrow$

$I + \Delta I$

10.518673

$$\Delta(\ln I) = -7.79615 \times 10^{-2}$$
$$= -0.0779615$$
$$= -7.79615\%$$

$$\Delta(\ln D) = -1.013500 \times 10^{-2}$$
$$= -0.01013500$$

new: $\ln D + \Delta(\ln D) \approx \cancel{8.88862}$ 9.787992

new $D + \Delta D \approx e^{9.787992}$

$$\approx \boxed{17,818}$$

Alternative:

$$\frac{37000-40000}{40000} = \frac{\Delta I}{I}$$

$$-0.075 = -7.5\%$$

$$\frac{\Delta D/D}{\Delta I/I} \approx \frac{dD/D}{dI/I} = 0.13$$

$$\Delta D/D \approx 0.13\,\Delta I/I$$

$$\approx -0.00975 = -0.975\%$$

$$\Delta D \approx -0.00975 \cdot \underset{18{,}000}{D} = -175.5$$

$$D + \Delta D \approx 18000 - 175.5 = 17{,}824.5$$

$$\left(\frac{xe^{3x}}{1+\sqrt{x}}\right)' = ? \quad f = \frac{xe^{3x}}{1+\sqrt{x}} \quad \ln f = \ln x + 3x - \ln(1+\sqrt{x})$$

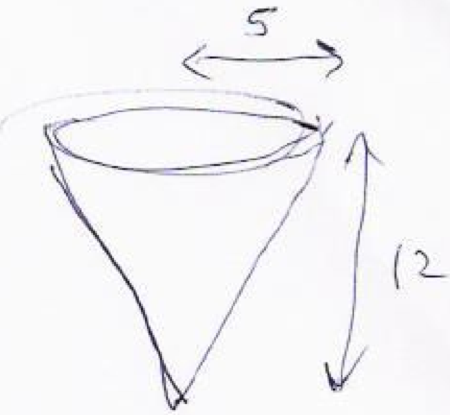
$$\frac{df}{f} = d(\ln f) = \frac{dx}{x} + 3dx - \frac{d(1+\sqrt{x})}{1+\sqrt{x}}$$

$$\frac{df}{f} = \frac{dx}{x} + 3dx - \frac{0 + dx/(2\sqrt{x})}{1+\sqrt{x}} = \boxed{\left(\frac{1}{x} + 3 - \frac{1}{2\sqrt{x}(1+\sqrt{x})}\right)dx}$$

$$\left(\frac{xe^{3x}}{1+\sqrt{x}}\right)' = f' = \frac{df}{dx} = \frac{df}{f} \cdot \frac{f}{dx} = \left(\frac{1}{x} + 3 - \frac{1}{2\sqrt{x}(1+\sqrt{x})}\right)f$$

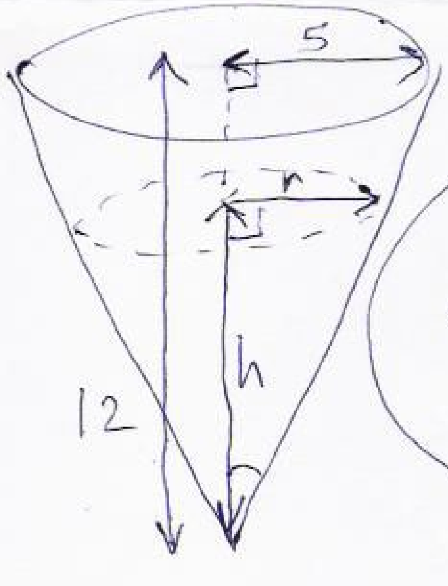
$$= \left(\frac{1}{x} + 3 - \frac{1}{2\sqrt{x}(1+\sqrt{x})}\right)\left(\frac{xe^{3x}}{1+\sqrt{x}}\right)$$

A conical paper cup has height 12 cm & radius 5 cm.



If the cup is being filled at a rate of $50\ cm^3/sec$ at the liquid in the cup is currently 7 cm deep, then how fast is depth increasing?

(Hint: $V_{cone} = \frac{1}{3}\pi r^2 h$)



$\frac{h}{r} = \frac{12}{5}$ always

$V_{liquid} = \frac{1}{3}\pi r^2 h$ always

$h = \frac{12}{5} r$

$V = \frac{1}{3}\pi r^2 \left(\frac{12}{5} r\right) = \frac{4\pi}{5} r^3$

$dV = \frac{4\pi}{5}(3r^2 dr)$

$$\frac{dV}{dt} = \frac{4\pi}{5}\left(3r^2 \frac{dr}{dt}\right) = \frac{12\pi}{5} r^2 \frac{dr}{dt}$$

Right now: $r = 7$, $\frac{dV}{dt} = 50$

$$50 = \frac{12\pi}{5}(7)^2 \frac{dr}{dt}$$

$250 = 12\pi(49)\, dr/dt$ But we

$0.1353 \approx dr/dt$ want dh/dt...

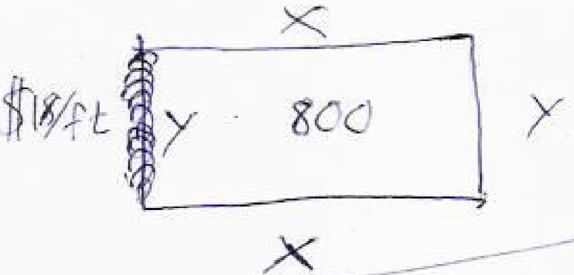
always $\frac{h}{r} = \frac{12}{5} \Rightarrow 5h = 12r \Rightarrow 5\,dh = 12\,dr$

$$\frac{dh}{dt} = \frac{12}{5}\frac{dr}{dt} \Leftarrow dh = \frac{12}{5} dr$$

0.3248 cm/sec is dh/dt

Optimization:

#25 (12-6) Build rectangular fence w/ area 800 ft^2 inside. 3 sides cost \$6/ft; the 4th side costs \$18/ft. Find the dimensions of the cheapest fence



Constraint:

$xy = 800$

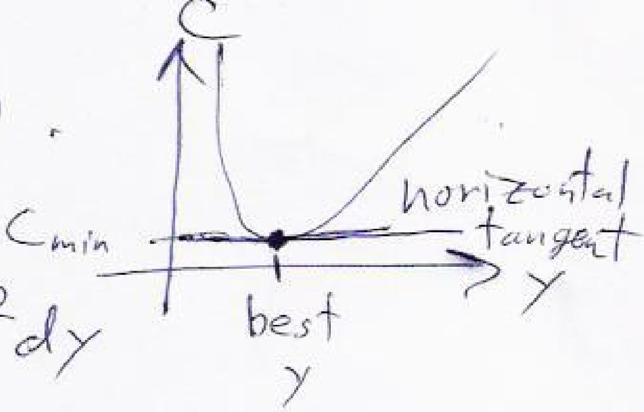
Minimize cost $C = 18y + 6(2x + y)$

Solve the constraint for y or x.

$x = \frac{800}{y} \rightarrow C = 18y + 6\left(\frac{1600}{y} + y\right)$

$C = 24y + \frac{9600}{y} = 24y + 9600y^{-1}$

Find where $\frac{dC}{dy} = 0$.



$dC = 24\,dy + 9600(-1)y^{-2}\,dy$

$dC/dy = 24 - 9600/y^2$

Solve $0 = 24 - 9600/y^2$

$\frac{9600}{y^2} = 24$

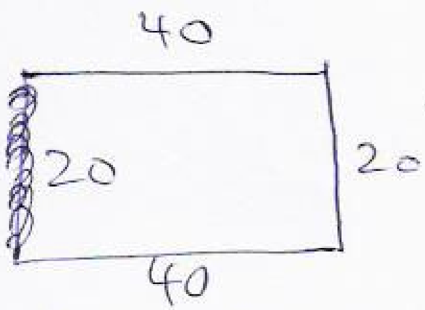
$9600 = 24y^2$

$400 = y^2$

$20 = y$

$x = \frac{800}{y}$

$x = \frac{800}{20} = 40$



$20' \times 40'$ with the expensive side $20'$ is best.