Topics

9/20 optimization ✓

9/22 derivatives: ✓

$\ln x$

e^x

9/27 elasticity

9/29 implicit diff. ✓

10/4] related rates
10/6]

Find $d\left(\sqrt{[\ln(x+e^x)]x}\right)$.

Show your steps.

This is the differential
of the square root
of the product
of x
and the logarithm

of the sum
of x
and the exponential
of x.

$$d(\sqrt{f}) = \frac{df}{2\sqrt{f}} =$$

$$d(f^n) = n f^{n-1} df$$

$$d(f^{1/2}) = \frac{1}{2} f^{-1/2} df$$

$$d\left(\sqrt{[\ln(x+e^x)]x}\right) = \frac{d([\ln(x+e^x)]x)}{2\sqrt{[\ln(x+e^x)]x}}$$

$$d(fg) = df \cdot g + f \cdot dg$$

$$\frac{d(\ln(x+e^x)) x + [\ln(x+e^x)] dx}{2\sqrt{[\ln(x+e^x)]x}} =$$

$$d(\ln f) = \frac{df}{f}$$

$$= \frac{\left[\left(d(x+e^{x})\right)/(x+e^{x})\right]x + [\ln(x+e^{x})]dx}{2\sqrt{[\ln(x+e^{x})]x}}$$

$$d(f+g) = df + dg$$

$$= \frac{\left[\left(dx + d(e^{x})\right)/(x+e^{x})\right]x + [\ln(x+e^{x})]dx}{2\sqrt{[\ln(x+e^{x})]x}}$$

$$d(e^{f}) = e^{f}\,df$$

$$= \frac{\left[(dx + e^{x}dx)/(x+e^{x})\right]x + [\ln(x+e^{x})]dx}{2\sqrt{[\ln(x+e^{x})]x}}$$

$$\left(\sqrt{[\ln(x+e^{x})]x}\right)' = \frac{\left[(1+e^{x})/(x+e^{x})\right]x + \ln(x+e^{x})}{2\sqrt{[\ln(x+e^{x})]x}}$$

$$d\left(\frac{e^{2x}}{x}\right) = \frac{d(e^{2x})x - e^{2x}\,dx}{x^{2}}$$

$$d\left(\frac{f}{g}\right) = \frac{df\cdot g - f\cdot dg}{g^{2}}$$

$$d\left(\frac{e^{2x}}{x}\right)=\frac{e^{2x}d(2x)\,x-e^{2x}dx}{x^2}$$

k constant $\Rightarrow d(kf)=k\,df$

$$d\left(\frac{e^{2x}}{x}\right)=\frac{2e^{2x}(dx)\,x-e^{2x}dx}{x^2}$$

$$\left(\frac{e^{2x}}{x}\right)'=\frac{2e^{2x}x-e^{2x}}{x^2}$$

Optimization:

Constraint $f(x,y)=k$

Maximize (or minimize) $g(x,y)$

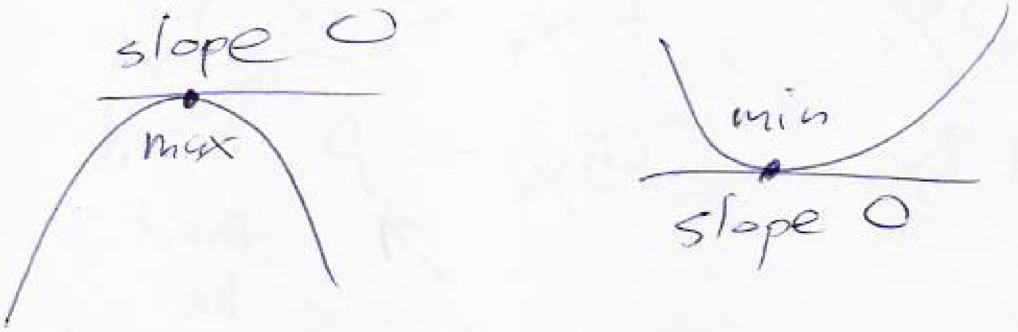
1. Solve $f(x,y)=k$ for x (or for y). to get $x=h(y)$ (or to get $y=h(x)$).

2. Now maximize (or minimize) $g(h(y),y)$ (or $g(x,h(x))$

(3) To find the max/min:

solve $(g(h(y), y))' = 0$

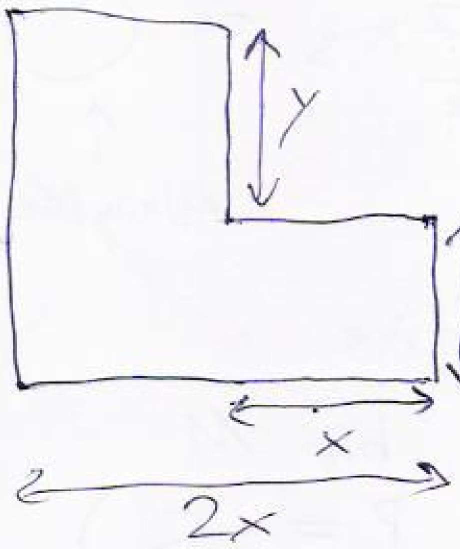
(or $(g(x, h(x)))') = 0$.



← Graph the function to confirm you found the max. (or min).

(4) If you found the y (or x) where the max. (or min) is, then plug that in to your formula ~~...~~ $x = h(y)$ to get the x and then plug x & y into $g(x,y)$ to find the max/min value.

A fence is to have area $500\ m^2$ and be in the shape of a square adjoined to rectangles to make an L:

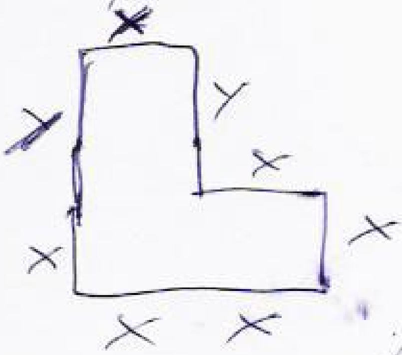


The bottom of the L should be twice as wide as the square.

Minimize the fence length.

Constraint: $x^2 + x(x+y) = 500$

Minimize $6x + 2y$



$$x(x+y) = 500 - x^2$$

$$x + y = \frac{500}{x} - x$$

$$y = \frac{500}{x} - 2x$$

Minimize $6x + 2\left(\frac{500}{x} - 2x\right)$

Solve $0 = \left(6x + \frac{1000}{x} - 4x\right)'$

$0 = \left(2x + \frac{1000}{x}\right)'$

$0 = \left(2x + 1000x^{-1}\right)'$

$d\left(\frac{1}{f}\right) = -\frac{df}{f^2}$ $\quad d(f^{-1}) = -1f^{-2}df$

$d(f^n) = nf^{n-1}df$

$0 = 2 + 1000(-1)x^{-2}$

$0 = 2 - 1000/x^2$

$\frac{1000}{x^2} = 2$

$1000 = 2x^2$

$500 = x^2$

$\pm\sqrt{500} = x$

$22.36\ldots = \sqrt{500} = x$ (x is a length)

Confirmed: $x = \sqrt{500}$ is where fence length is minimized.

$y = \frac{500}{\sqrt{500}} - 2\sqrt{500} = -\sqrt{500}$ ← not possible

Need $y \geq 0$

Need $\frac{500}{x} - 2x \geq 0$

$$\frac{500}{x} \geq 2x$$

$$500 \geq 2x^2 \qquad (x \geq 0)$$

$$250 \geq x^2$$

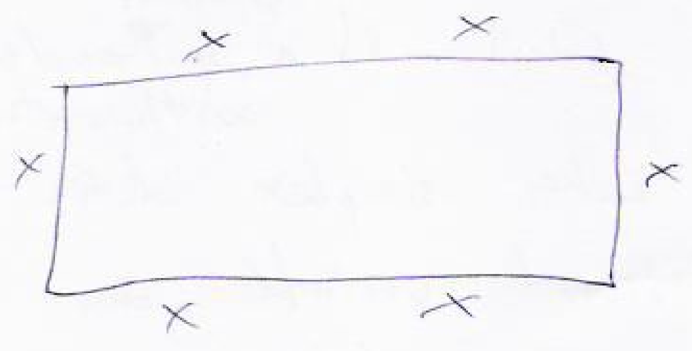
$$\sqrt{250} \geq x \geq 0$$

$0 = \left(6x + \frac{1000}{x} - 4x\right)'$ has no solutions except $x = \pm\sqrt{500}$, neither satisfies $\sqrt{250} \geq x \geq 0$.

The minimum must be at $x = 0$ or $x = \sqrt{250} \approx 15.8$

It's at $x = \sqrt{250}$

$$y = \frac{500}{\sqrt{250}} - 2\sqrt{250} = 0:$$

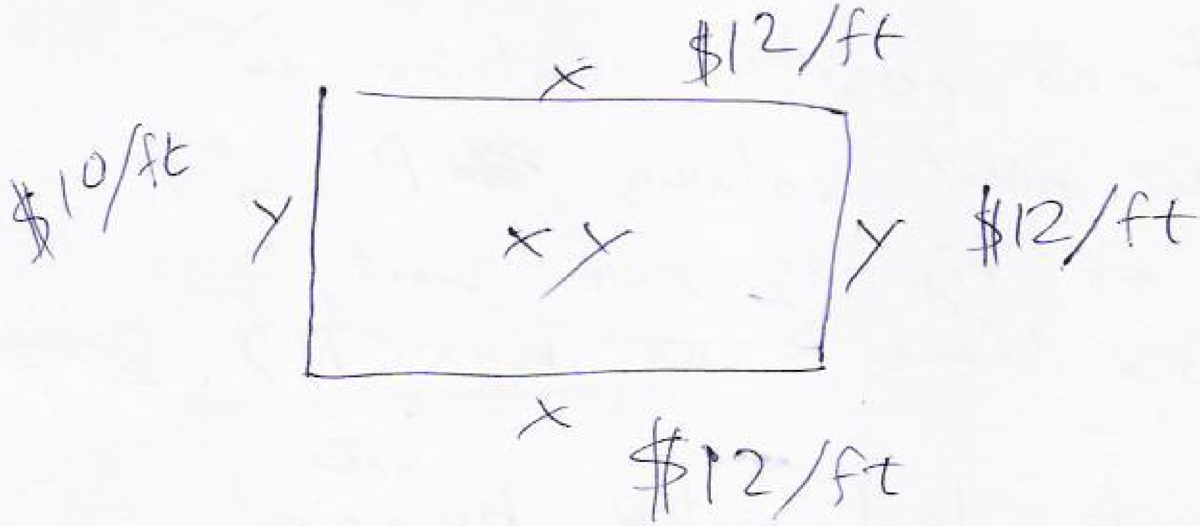


min. length $= 6x + 2y = 6\sqrt{250} + 0$ ✓

One side of rectangular fence cost \$10/ft.

The other sides cost \$12/ft.

Given \$30,000, maximize enclosed area.



$30{,}000 = C = 10y + 12(2x+y)$

Maximize $xy = A$

$30000 - 10y = 12(2x+y)$

$\frac{30000 - 10y}{12} = 2x + y$

$$\frac{30000-10y}{12} - y = 2x$$

$$\frac{15000-5y}{12} - \frac{y}{2} = x$$

$$\frac{15000-5y-6y}{12} = x$$

$$1250 - 11y/12 = x$$

$$A = xy = \left(1250 - \frac{11}{12}y\right)y$$

$$A = 1250y - \frac{11}{12}y^2$$

$$A' = 1250 - \frac{11}{12}(2y)$$

Solve $0 = A'$:

$$\frac{11}{6}y = \frac{11}{12}(2y) = 1250$$

$$y = \frac{6}{11} \cdot 1250 \approx 681.8$$

$$x = 1250 - \frac{11}{12}y = 1250 - \frac{1}{2}1250$$

$$x = 625$$

$$A = xy = 625 \cdot \frac{6}{11} \cdot 1250 \approx 426{,}136\ ft^2$$

Consider the curve

$$x^2 \ln y + x = 12 + y - e$$

Find ~~the~~ an equation for the line tangent to this curve at the point

$$(x, y) = (3, e)$$

$$d(x^2 \ln y + x) = d(12 + y - e)$$

$$d(x^2) \ln y + x^2 d(\ln y) + dx$$

$$= 0 + dy - 0$$

$$2x\,dx \ln y + x^2\,dy/y + dx = dy$$

Plug in $(x, y) = (3, e)$

$$6\,dx \cdot 1 + 9\,dy/e + dx = dy$$

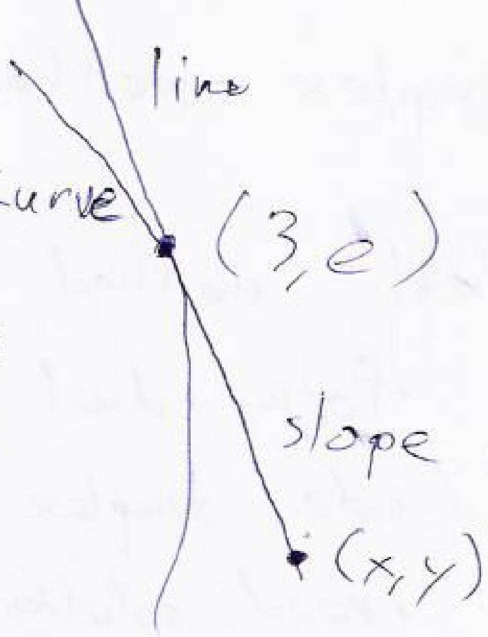
$$6 + \frac{9\,dy}{e\,dx} + 1 = \frac{dy}{dx}$$

$m = \text{slope} = \frac{dy}{dx}$ $\quad 6 + \frac{9}{e} m + 1 = m$

$$7 = \left(1 - \frac{9}{e}\right) m = \frac{e-9}{e} m$$

$$\frac{7e}{e-9} = m \approx -3.0291$$

Tangent line



$$m \doteq \frac{\Delta y}{\Delta x} = \frac{y-e}{x-3}$$

$$y - e = m(x-3)$$

$$y - e = \frac{7e}{e-9}(x-3)$$

Implicit diff:

Compute $d(\ldots)$ of both sides $\_ = \_$

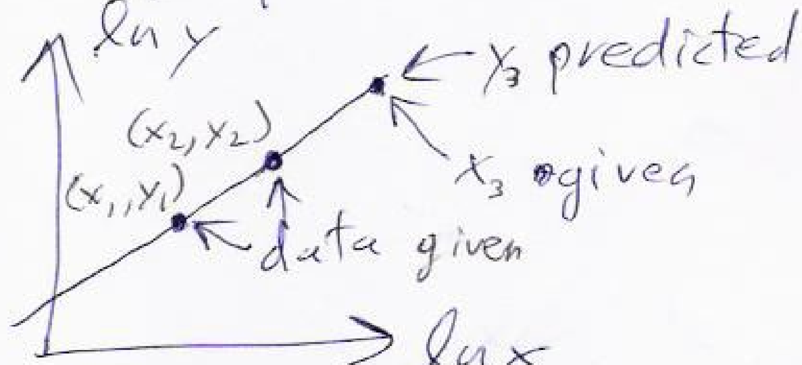
Plug in your point $(x, y) = (a, b)$

Solve for dy/dx to get slope m.

Line: $y - b = m(x-a)$.
(tangent)

Elasticity: The x-elasticity of y of $\frac{dy/y}{dx/x} = \frac{d(\ln y)}{d(\ln x)}$

If elasticity is constant, then the $\ln y - \ln x$ graph is a line.



2 data points determine the line, allows ~~you~~ you to make prediction.

Related rates:

Usually, you have to find the absolute or relative rate of change of something. (Assume absolute unless otherwise specified.)

$\frac{dx}{dt}$ = (absolute) rate of change of x (t = time)

$$\frac{dx/x}{dt} = \frac{1}{x}\frac{dx}{dt} = \frac{d(\ln x)}{dt}$$

is the relative rate of change

$\frac{1}{x}\frac{dx}{dt} \cdot 100$ is the percentage rate of change.

E = pop. of Europe (in millions)

$E = 595$ ← right now

$$\frac{1}{E} \cdot \frac{dE}{dt} = 0.223\,\% = 0.00223$$

$$100 \cdot \frac{1}{E}\frac{dE}{dt} = 0.223$$

G = pop. of Germany (in millions

$G = 82.1$ ← right now

$$\frac{1}{G}\frac{dG}{dt} = -0.0952\,\% = -0.000952$$

What is the relative rate of change of the non-German European population right now?

$$\frac{1}{E-G}\underbrace{\frac{d(E-G)}{dt}}_{\text{absolute rate of change}} = \text{relative rate of change}$$

$$\frac{1}{E-G}\left(\frac{dE}{dt}-\frac{dG}{dt}\right)$$

right now: $\longrightarrow = \frac{1}{E-G}\left(.00223E-(-.000952G)\right)$

$$= \frac{1}{595-82.1}\left(.00223(595)+.000952(82.1)\right)$$

$$\approx \boxed{0.00274 = 0.274\%}$$

With related rates,
write down formulas
true all the time.

Take d(...) of both sides;
divide by dt.
Plug in your data for right now.
Find the rates you want.

~~Or, if you have a formula~~

(This is not the only strategy.)

E Europe
G German
N non-German

$$E = G + N \quad \text{always}$$

$$dE = dG + dN$$

$$\frac{dE}{dt} = \frac{dG}{dt} + \frac{dN}{dt}$$

Find $\frac{1}{N}\frac{dN}{dt}$ ...

Right now:

$$0.00223E = -.000952G + \frac{dN}{dt}$$

$E = 595 \quad G = 82.1$

$595 = 82.1 + N$

$E = G + N$

$N = 512.9$

$$0.00223(595) + .000952(82.1) = \frac{dN}{dt}$$

$$\boxed{\frac{0.00223(595) + .000952(82.1)}{512.9}} = \frac{1}{N}\frac{dN}{dt}$$

≈ 0.00274

A sphere $(V = \frac{4}{3}\pi r^3)$

is expanding at $2\,cm^3/s$.

What is the rate of change

of its diameter $(D = 2r)$

when the diameter is $3\,cm$?

Always: $V = \frac{4}{3}\pi r^3$

$D = 2r$

$dV = \frac{4}{\cancel{3}}\pi \cdot \cancel{3} r^2\, dr$

$dD = 2\,dr$

$dV/dt = 4\pi r^2 \frac{dr}{dt}$

$dD/dt = 2\frac{dr}{dt}$

Right now: $D = 3 = 2r \Rightarrow r = 1.5$

$2 = dV/dt = 4\pi r^2 \frac{dr}{dt}$

$2 = 4\pi(1.5)^2\, dr/dt$

$$2 = 9\pi \, dr/dt$$

$$\frac{2}{9\pi} = \frac{dr}{dt}$$

$$\frac{dD}{dt} = 2\frac{dr}{dt} = \boxed{\frac{4}{9\pi} \text{ cm/s}}$$