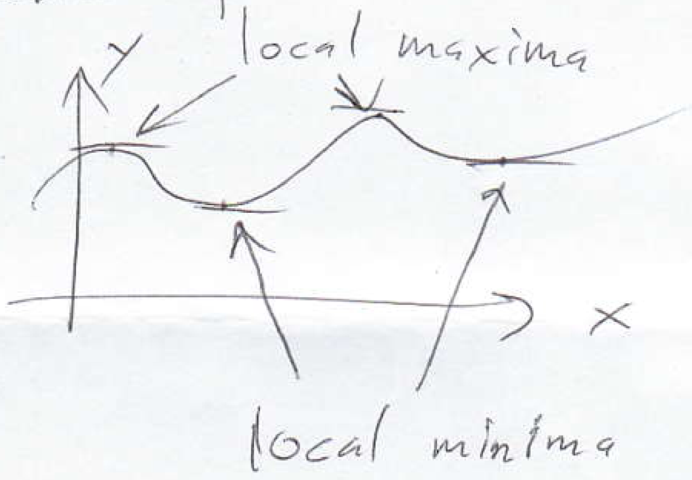
# 2-variable optimization (15-3)

Recall 1-variable optimization:

$y = f(x)$




$f'(x) \neq 0 \Rightarrow$ not local extrema

$$\begin{cases} f'(x) > 0 & \nearrow \\ f'(x) < 0 & \searrow \end{cases}$$

---

$f'(x) = 0$ & $f''(x) > 0$

$\Rightarrow$ local ~~maximum~~ minimum


---

$f'(x) = 0$ & $f''(x) < 0$

$\Rightarrow$ local maximum


---

$f'(x) = 0$ & $f''(x) = 0 \Rightarrow$ ?

$z = f(x,y)$

$$\frac{\partial f}{\partial x} \neq 0 \Rightarrow \begin{cases} \nearrow \text{ or } \searrow \text{ as } \begin{cases} y \text{ fixed} \\ x \ \cancel{\text{changes}} \nearrow \end{cases} \\ \text{not local extrema} \end{cases}$$

$$\frac{\partial f}{\partial y} \neq 0 \Rightarrow \begin{cases} \text{not local extrema} \\ \nearrow \text{ or } \searrow \text{ as } \begin{cases} x \text{ fixed} \\ y \ \nearrow \end{cases} \end{cases}$$

Local extrema only happens where

$$\frac{\partial f}{\partial x} = \frac{\partial f}{\partial y} = 0.$$

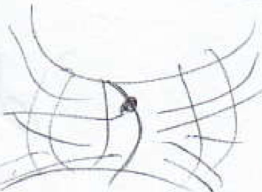
page 8.17

$$\frac{\partial f}{\partial x} = \frac{\partial f}{\partial y} = 0 \ \& \ \underbrace{\frac{\partial^2 f}{\partial x^2}\frac{\partial^2 f}{\partial y^2} - \left(\frac{\partial^2 f}{\partial x \partial y}\right)^2}_{\text{discriminant}} > 0 \ \& \ \frac{\partial^2 f}{\partial x^2} > 0$$

$\Rightarrow$ local min.

min.

$$\frac{\partial f}{\partial x} = \frac{\partial f}{\partial y} = 0 \ \& \ \frac{\partial^2 f}{\partial x^2}\frac{\partial^2 f}{\partial y^2} - \left(\frac{\partial^2 f}{\partial x \partial y}\right)^2 < 0$$



saddle point

$$\frac{\partial f}{\partial x} = \frac{\partial f}{\partial x} = 0 \ \& \ \frac{\partial^2 f}{\partial x^2}\frac{\partial^2 f}{\partial y^2} - \left(\frac{\partial^2 f}{\partial x \partial y}\right)^2 > 0$$

$$\& \ \frac{\partial^2 f}{\partial x^2} < 0 \Rightarrow \text{local max}$$

max

$$\frac{\partial f}{\partial x} = \frac{\partial f}{\partial y} = 0 = \frac{\partial^2 f}{\partial x^2}\frac{\partial^2 f}{\partial y^2} - \left(\frac{\partial^2 f}{\partial x \partial y}\right)^2 \Rightarrow ?$$

$$f'(x) = \frac{df}{dx} \quad \& \quad f''(x) = \frac{d(df)}{(dx)^2}$$

$$f''(x) = \frac{d^2f}{dx^2}$$

---

$$df = d(f(x,y)) = \frac{\partial f}{\partial x}dx + \frac{\partial f}{\partial y}dy$$

$$\frac{\partial f}{\partial x} = \frac{df}{dx} \quad \text{when } \underbrace{dy = 0}$$

same as treating $y$ as a constant

$$\frac{\partial f}{\partial y} = \frac{df}{dy} \quad \text{when } \underbrace{dx = 0}$$

same as treating $x$ as a const.

$$d^2f = d(df) = \frac{\partial^2 f}{\partial x^2}dx^2 + 2\frac{\partial^2 f}{\partial x \partial y}dx\,dy + \frac{\partial^2 f}{\partial y^2}dy^2$$

$$\frac{\partial^2 f}{\partial x^2} = \frac{d(\partial f/\partial x)}{dx} \quad \text{when } \underbrace{dy = 0}_{y \text{ const}}$$

$$\frac{\partial^2 f}{\partial y^2} = \frac{d(\partial f/\partial y)}{dy} \quad \text{when } \underbrace{dx = 0}_{x \text{ const.}}$$

$$\frac{\partial^2 f}{\partial x \partial y} = \frac{d(\partial f/\partial y)}{dx} \quad \text{when } \underbrace{dy = 0}_{y \text{ const}}$$

$$\frac{\partial^2 f}{\partial y \partial x} = \frac{\partial^2 f}{\partial x \partial y}$$

$$= \frac{d(\partial f/\partial x)}{dy} \quad \text{when } \underbrace{dx = 0}_{x \text{ const}}$$

$$f(x,y) = x^2 + xy + 3y^2$$

$$\frac{\partial f}{\partial x} = ? \qquad \frac{\partial f}{\partial y} = ?$$

$$df = 2x\,dx + dx \cdot y + x \cdot dy + 3(2y\,dy)$$

$$df = (2x + y)\,dx + (x + 6y)\,dy$$

$$\frac{\partial f}{\partial x} = 2x + y \qquad \frac{\partial f}{\partial y} = x + 6y$$

Or: let $y$ be constant:

$$f_x = \frac{\partial f}{\partial x} = \frac{df}{dx} = \frac{2x\,dx + dx \cdot y + 0}{dx}$$

$$= 2x + y$$

let $x$ be constant:

$$f_y = \frac{\partial f}{\partial y} = \frac{df}{dy} = \frac{0 + x \cdot dy + 3(2y\,dy)}{dy}$$

$$= x + 6y$$

$$\frac{\partial^2 f}{\partial x^2} = \frac{d(\partial f/\partial x)}{dx} \quad \text{with } y \text{ constant}$$

$$= \frac{2\,dx + 0}{dx} = 2$$

$$\frac{\partial^2 f}{\partial y^2} = \frac{d(\partial f/\partial y)}{dy} \quad \text{with } x \text{ constant}$$

$$= \frac{0 + 6\,dy}{dy} = 6$$

$$\frac{\partial^2 f}{\partial x\,\partial y} = \frac{d(\partial f/\partial y)}{dx} \quad \text{with } y \text{ constant}$$

$$= \frac{dx + 0}{dx} = 1$$

same

$$\frac{\partial^2 f}{\partial y\,\partial x} = \frac{d(\partial f/\partial x)}{dy} \quad \text{with } x \text{ constant}$$

$$= \frac{0 + dy}{dy} = 1$$

$$f(x,y) = x^2 + xy + 3y^2$$

$$f_x = \frac{\partial f}{\partial x} = 2x + y \qquad f_y = x + 6y = \frac{\partial f}{\partial y}$$

$$f_{xx} = \frac{\partial^2 f}{\partial x^2} = 2 \qquad f_{yy} = \frac{\partial^2 f}{\partial y^2} = 6$$

$$f_{yx} = \frac{\partial^2 f}{\partial x \partial y} = 1 = \frac{\partial^2 f}{\partial y \partial x} = f_{xy}$$

Optimizing $f$:

1. Find where $\frac{\partial f}{\partial x} = \frac{\partial f}{\partial y} = 0$.
2. Check if $\frac{\partial^2 f}{\partial x^2}\frac{\partial^2 f}{\partial y^2} - \left(\frac{\partial^2 f}{\partial y \partial x}\right)^2 > 0$ at those points.
3. Where discriminant is $> 0$, (and $\partial f/\partial x = \partial f/\partial y = 0$), if $\partial^2 f/\partial x^2 > 0$, then local min; if $\partial^2 f/\partial x^2 < 0$, then local max.

(1) Where does $2x+y = \underbrace{x+6y}_{} = 0$?

$2x+y=0 \Rightarrow y=-2x$

$x+6(-2x)=0$

$-11x=0 \Rightarrow x=0 \Rightarrow y=-2(0)=0$

Only solution: $(x,y)=(0,0)$.

(2) Discriminant:

$$\frac{\partial^2 f}{\partial x^2}\frac{\partial^2 f}{\partial y^2}-\left(\frac{\partial^2 f}{\partial y \partial x}\right)^2 = 2\cdot 6 - 1^2 = 11$$

at $(x,y)=(0,0)$ $\qquad 11>0$

not saddle point

(3) $\frac{\partial^2 f}{\partial x^2}=2>0 \Rightarrow \boxed{f(0,0)=0 \text{ is a local min.}}$

at $(x,x)=(0,0)$

$$f(x,y) = x^3 + xy + 3y^2 + 5$$

$$\frac{\partial f}{\partial x} = \frac{df}{dx} \text{ with } y \text{ constant}$$

$$= \frac{3x^2dx + dx \cdot y + 0 + 0}{dx} = 3x^2 + y$$

$$\frac{\partial f}{\partial y} = \frac{df}{dy} \text{ with } x \text{ constant}$$

$$\frac{0 + x \cdot dy + 3(2y\,dy) + 0}{dy} = x + 6y$$

① Solve $\frac{\partial f}{\partial x} = \frac{\partial f}{\partial y} = 0$.

$$3x^2 + y = 0 = x + 6y$$

$$\Downarrow$$

$$y = -3x^2$$

$$0 = x + 6(-3x^2)$$

$$0 = x - 18x^2$$

$$y = -3(0)^2 = 0$$

$$\uparrow$$

$x = 0$ or

$1 - 18x = 0 \Leftarrow 0 = x(1 - 18x)$

$\hookrightarrow 1 = 18x$

$1/18 = x \rightarrow y = -3\left(\frac{1}{18}\right)^2 = -\frac{1}{108}$

Two solutions: $(x,y) = (0,0)$

$(x,y) = \left(\frac{1}{18}, -\frac{1}{108}\right)$

$3x^2 + y$

Check:

| | | |
|---|---|---|
| $\partial f/\partial x$ | 0 ✓ | 0 ✓ |
| $\partial f/\partial y$ | 0 ✓ | 0 ✓ |
| $(x,y)$ | $(0,0)$ | $\left(\frac{1}{18}, -\frac{1}{108}\right)$ |

$x + 6y$

(2) Check if discriminant $> 0$ at these points.

$$\text{disc.} = \frac{\partial^2 f}{\partial x^2}\frac{\partial^2 f}{\partial y^2} - \left(\frac{\partial^2 f}{\partial y \partial x}\right)^2$$

$$\frac{\partial^2 f}{\partial x^2} = \frac{d(\partial f/\partial x)}{dx} \text{ with } y \text{ constant}$$

$$= \frac{d(3x^2+y)}{dx} = \frac{3(2x\,dx) + 0}{dx} = 6x$$

$$\frac{\partial^2 f}{\partial y^2} = \frac{d(\partial f/\partial y)}{dy} \text{ with } x \text{ constant}$$

$$= \frac{d(x+6y)}{dy} = \frac{0 + 6dy}{dy} = 6$$

$$\frac{\partial^2 f}{\partial y \partial x} = \frac{d(\partial f/\partial x)}{dy} \quad \text{with } x \text{ constant.}$$

$$= \frac{d(3x^2+y)}{dy} = \frac{0+dy}{dy} = 1$$

$$\text{disc.} = 6x \cdot 6 - 1^2 = 36x - 1$$

At $(x,y)=(0,0)$, disc. $= -1 < 0$ $\Downarrow$ saddle point

At $(x,y)=\left(\frac{1}{18}, -\frac{1}{108}\right)$ disc $= \frac{36}{18} - 1 = 1 > 0$ $\Downarrow$ not saddle pt.

$$\frac{\partial^2 f}{\partial x^2} = 6x = 6\left(\frac{1}{18}\right) = \frac{1}{3} > 0$$

$\Downarrow$

$f\left(\frac{1}{18}, -\frac{1}{108}\right) = 4.99991\ldots$ is local min.

HW (15-3) #15, 16, 21