22 Find the derivative of $f(x) = 2x^2$ at the point $x = 3$.

23 Find the slope of the curve $f(x) = \sqrt{x - 1}$ at the point $x = 5$.

24 An object moves according to the equation $y = 1/(t + 2)$, $t \geq 0$. Find the velocity as a function of t.

25 A particle moves according to the equation $y = t^4$. Find the velocity as a function of t.

26 Suppose the population of a town grows according to the equation $y = 100t + t^2$. Find the rate of growth at time $t = 100$ years.

27 Suppose a company makes a total profit of $1000x - x^2$ dollars on x items. Find the marginal profit in dollars per item when $x = 200$, $x = 500$, and $x = 1000$.

28 Find the derivative of the function $f(x) = |x + 1|$.

29 Find the derivative of the function $f(x) = |x^3|$.

30 Find the slope of the parabola $y = ax^2 + bx + c$ where a, b, c are constants.

2.2 DIFFERENTIALS AND TANGENT LINES

Suppose we are given a curve $y = f(x)$ and at a point (a, b) on the curve the slope $f'(a)$ is defined. Then the *tangent line* to the curve at the point (a, b), illustrated in Figure 2.2.1, is defined to be the straight line which passes through the point (a, b) and has the same slope as the curve at $x = a$. Thus the tangent line is given by the equation

$$l(x) - b = f'(a)(x - a),$$

or

$$l(x) = f'(a)(x - a) + b.$$

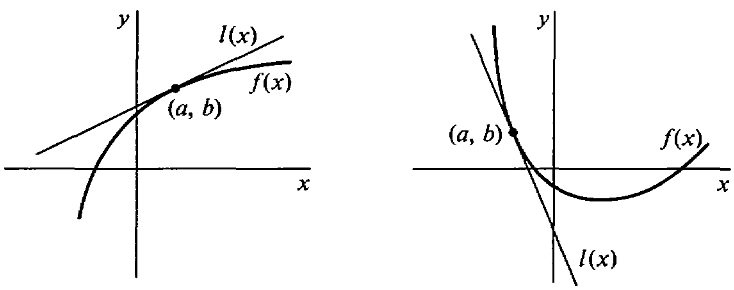

Figure 2.2.1 Tangent lines.

EXAMPLE 1 For the curve $y = x^3$, find the tangent lines at the points $(0, 0)$, $(1, 1)$, and $(-\frac{1}{2}, -\frac{1}{8})$ (Figure 2.2.2).

The slope is given by $f'(x) = 3x^2$. At $x = 0$, $f'(0) = 3 \cdot 0^2 = 0$. The tangent line has the equation

$$y = 0(x - 0) + 0, \quad \text{or} \quad y = 0.$$

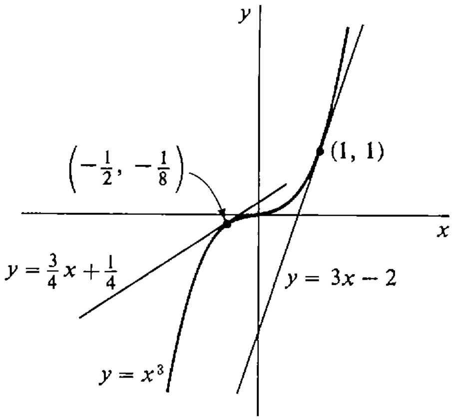

Figure 2.2.2

At $x = 1, f'(1) = 3$, whence the tangent line is

$$y = 3(x - 1) + 1, \quad \text{or} \quad y = 3x - 2.$$

At $x = -\frac{1}{2}, f'(-\frac{1}{2}) = 3 \cdot (-\frac{1}{2})^2 = \frac{3}{4}$, so the tangent line is

$$y = \tfrac{3}{4}(x - (-\tfrac{1}{2})) + (-\tfrac{1}{8}), \quad \text{or} \quad y = \tfrac{3}{4}x + \tfrac{1}{4}.$$

Given a curve $y = f(x)$, suppose that x starts out with the value a and then changes by an infinitesimal amount Δx. What happens to y? Along the curve, y will change by the amount

$$f(a + \Delta x) - f(a) = \Delta y.$$

But along the tangent line y will change by the amount

$$\begin{aligned} l(a + \Delta x) - l(a) &= [f'(a)(a + \Delta x - a) + b] - [f'(a)(a - a) + b] \\ &= f'(a)\,\Delta x. \end{aligned}$$

When x changes from a to $a + \Delta x$, we see that:

$$\begin{aligned} \text{change in } y \text{ along curve} &= f(a + \Delta x) - f(a), \\ \text{change in } y \text{ along tangent line} &= f'(a)\,\Delta x. \end{aligned}$$

In the last section we introduced the dependent variable Δy, the increment of y, with the equation

$$\Delta y = f(x + \Delta x) - f(x).$$

Δy is equal to the change in y along the curve as x changes to $x + \Delta x$.

The following theorem gives a simple but useful formula for the increment Δy.

INCREMENT THEOREM

Let $y = f(x)$. Suppose $f'(x)$ exists at a certain point x, and Δx is infinitesimal. Then Δy is infinitesimal, and

$$\Delta y = f'(x)\,\Delta x + \varepsilon\,\Delta x$$

for some infinitesimal ε, which depends on x and Δx.

PROOF

Case 1 $\Delta x = 0$. In this case, $\Delta y = f'(x)\,\Delta x = 0$, and we put $\varepsilon = 0$.

Case 2 $\Delta x \neq 0$. Then

$$\frac{\Delta y}{\Delta x} \approx f'(x);$$

so for some infinitesimal ε,

$$\frac{\Delta y}{\Delta x} = f'(x) + \varepsilon.$$

Multiplying both sides by Δx,

$$\Delta y = f'(x)\,\Delta x + \varepsilon\,\Delta x.$$

EXAMPLE 2 Let $y = x^3$, so that $y' = 3x^2$. According to the Increment Theorem,

$$\Delta y = 3x^2\,\Delta x + \varepsilon\,\Delta x$$

for some infinitesimal ε. Find ε in terms of x and Δx when $\Delta x \neq 0$. We have

$$\Delta y = 3x^2\,\Delta x + \varepsilon\,\Delta x,$$

$$\frac{\Delta y}{\Delta x} = 3x^2 + \varepsilon,$$

$$\varepsilon = \frac{\Delta y}{\Delta x} - 3x^2.$$

We must still eliminate Δy. From Example 1 in Section 2.1,

$$\Delta y = (x + \Delta x)^3 - x^3,$$

$$\frac{\Delta y}{\Delta x} = 3x^2 + 3x\,\Delta x + (\Delta x)^2.$$

Substituting, $\varepsilon = (3x^2 + 3x\,\Delta x + (\Delta x)^2) - 3x^2.$

Since $3x^2$ cancels,

$$\varepsilon = 3x\,\Delta x + (\Delta x)^2.$$

We shall now introduce a new dependent variable dy, called the differential of y, with the equation

$$dy = f'(x)\,\Delta x.$$

dy is equal to the change in y along the tangent line as x changes to $x + \Delta x$. In Figure 2.2.3 we see dy and Δy under the microscope.

Δy = change in y along curve
dy = change in y along tangent line

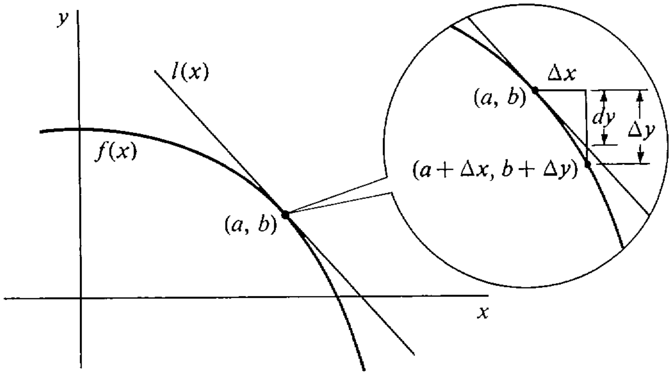

Figure 2.2.3

To keep our notation uniform we also introduce the symbol dx as another name for Δx. For an independent variable x, Δx and dx are the same, but for a dependent variable y, Δy and dy are different.

DEFINITION

Suppose y depends on x, $y = f(x)$.

(i) *The **differential of** x is the independent variable $dx = \Delta x$.*
(ii) *The **differential of** y is the dependent variable dy given by*

$$dy = f'(x)\,dx.$$

When $dx \neq 0$, the equation above may be rewritten as

$$\frac{dy}{dx} = f'(x).$$

Compare this equation with

$$\frac{\Delta y}{\Delta x} \approx f'(x).$$

The quotient dy/dx is a very convenient alternative symbol for the derivative $f'(x)$. In fact we shall write the derivative in the form dy/dx most of the time.

The differential dy depends on two independent variables x and dx. In functional notation,

$$dy = df(x, dx)$$

where df is the real function of two variables defined by

$$df(x, dx) = f'(x)\,dx.$$

When dx is substituted for Δx and dy for $f'(x)\,dx$, the Increment Theorem takes the short form

$$\Delta y = dy + \varepsilon\,dx.$$

The Increment Theorem can be explained graphically using an infinitesimal microscope. Under an infinitesimal microscope, a line of length Δx is magnified to a line of unit length, but a line of length $\varepsilon\,\Delta x$ is only magnified to an infinitesimal length ε. Thus the Increment Theorem shows that when $f'(x)$ exists:

(1) The differential dy and the increment $\Delta y = dy + \varepsilon\,dx$ are so close to each other that they cannot be distinguished under an infinitesimal microscope.
(2) The curve $y = f(x)$ and the tangent line at (x, y) are so close to each other that they cannot be distinguished under an infinitesimal microscope; both look like a straight line of slope $f'(x)$.

Figure 2.2.3 is not really accurate. The curvature had to be exaggerated in order to distinguish the curve and tangent line under the microscope. To give an accurate picture, we need a more complicated figure like Figure 2.2.4, which has a second infinitesimal microscope trained on the point $(a + \Delta x, b + \Delta y)$ in the field of view of the original microscope. This second microscope magnifies $\varepsilon\,dx$ to a unit length and magnifies Δx to an infinite length.

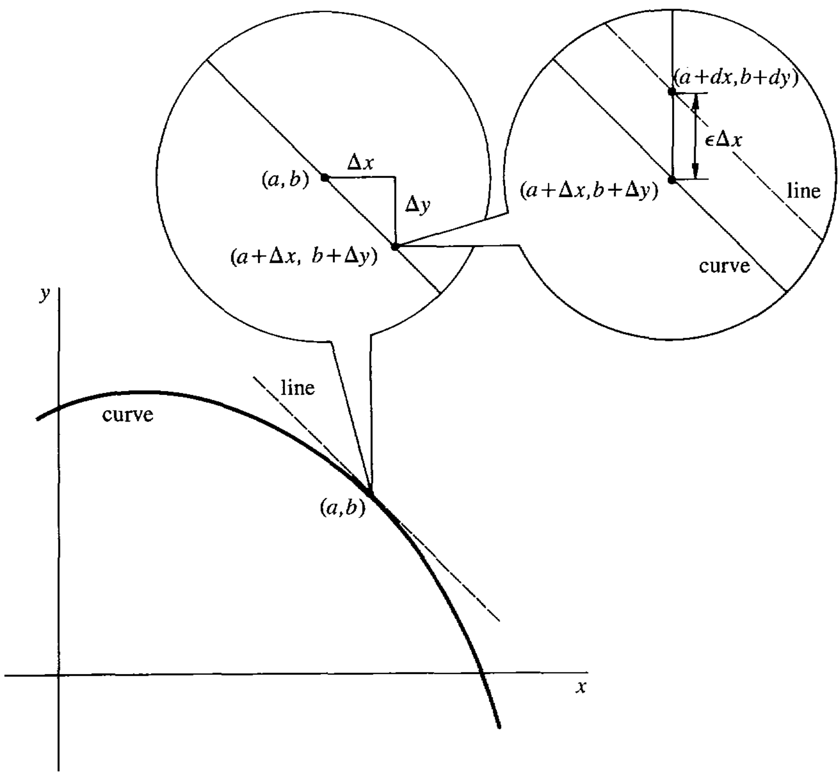

Figure 2.2.4

EXAMPLE 3 Whenever a derivative $f'(x)$ is known, we can find the differential dy at once by simply multiplying the derivative by dx, using the formula $dy = f'(x)\,dx$. The examples in the last section give the following differentials.

(a) $y = x^3$, $dy = 3x^2\,dx$.

(b) $y = \sqrt{x}$, $dy = \dfrac{dx}{2\sqrt{x}}$ where $x > 0$.

(c) $y = 1/x$, $dy = -dx/x^2$ when $x \neq 0$.

(d) $y = |x|$, $dy = \begin{cases} dx & \text{when } x > 0, \\ -dx & \text{when } x < 0, \\ \text{undefined} & \text{when } x = 0. \end{cases}$

(e) $y = bt - 16t^2$, $dy = (b - 32t)\,dt$.

The differential notation may also be used when we are given a system of formulas in which two or more dependent variables depend on an independent variable. For example if y and z are functions of x,

$$y = f(x), \qquad z = g(x),$$

then Δy, Δz, dy, dz are determined by

$$\Delta y = f(x + \Delta x) - f(x), \qquad \Delta z = g(x + \Delta x) - g(x),$$
$$dy = f'(x)\,dx, \qquad dz = g'(x)\,dx.$$

EXAMPLE 4 Given $y = \frac{1}{2}x$, $z = x^3$, with x as the independent variable, then

$$\Delta y = \tfrac{1}{2}(x + \Delta x) - \tfrac{1}{2}x = \tfrac{1}{2}\,\Delta x,$$
$$\Delta z = 3x^2\,\Delta x + 3x(\Delta x)^2 + (\Delta x)^3,$$
$$dy = \tfrac{1}{2}\,dx, \qquad dz = 3x^2\,dx.$$

The meaning of the symbols for increment and differential in this example will be different if we take y as the independent variable. Then x and z are functions of y.

$$x = 2y, \qquad z = 8y^3.$$

Now $\Delta y = dy$ is just an independent variable, while

$$\begin{aligned} \Delta x &= 2(y + \Delta y) - 2y = 2\,\Delta y, \\ \Delta z &= 8(y + \Delta y)^3 - 8y^3 \\ &= 8[3y^2\,\Delta y + 3y(\Delta y)^2 + (\Delta y)^3] \\ &= 24y^2\,\Delta y + 24y(\Delta y)^2 + 8(\Delta y)^3. \end{aligned}$$

Moreover, $dx = 2\,dy, \qquad dz = 24y^2\,dy.$

We may also apply the differential notation to terms. If $\tau(x)$ is a term with the variable x, then $\tau(x)$ determines a function f,

$$\tau(x) = f(x),$$

and the differential $d(\tau(x))$ has the meaning

$$d(\tau(x)) = f'(x)\,dx.$$

EXAMPLE 5

(a) $$d(x^3) = 3x^2\,dx.$$

(b) $$d(\sqrt{x}) = \frac{dx}{2\sqrt{x}}, \qquad x > 0.$$

(c) $$d(1/x) = -\frac{dx}{x^2}, \qquad x \neq 0.$$

(d) $$d(|x|) = \begin{cases} dx & \text{when } x > 0, \\ -dx & \text{when } x < 0, \\ \text{undefined} & \text{when } x = 0. \end{cases}$$

(e) Let $u = bt$ and $w = -16t^2$. Then

$$u + w = bt - 16t^2, \qquad d(u + w) = (b - 32t)\,dt.$$

PROBLEMS FOR SECTION 2.2

In Problems 1–8, express Δy and dy as functions of x and Δx, and for Δx infinitesimal find an infinitesimal ε such that $\Delta y = dy + \varepsilon\,\Delta x$.

1 $y = x^2$ **2** $y = -5x^2$

3 $y = 2\sqrt{x}$ **4** $y = x^4$

5 $y = 1/x$ **6** $y = x^{-2}$

7 $y = x - 1/x$ **8** $y = 4x + x^3$

9 If $y = 2x^2$ and $z = x^3$, find Δy, Δz, dy, and dz.

10 If $y = 1/(x + 1)$ and $z = 1/(x + 2)$, find Δy, Δz, dy, and dz.

11 Find $d(2x + 1)$ **12** Find $d(x^2 - 3x)$

13 Find $d(\sqrt{x + 1})$ **14** Find $d(\sqrt{2x + 1})$

15 Find $d(ax + b)$ **16** Find $d(ax^2)$

17 Find $d(3 + 2/x)$ **18** Find $d(x\sqrt{x})$

19 Find $d(1/\sqrt{x})$ **20** Find $d(x^3 - x^2)$

21 Let $y = \sqrt{x}$, $z = 3x$. Find $d(y + z)$ and $d(y/z)$.

22 Let $y = x^{-1}$ and $z = x^3$. Find $d(y + z)$ and $d(yz)$.

In Problems 23–30 below, find the equation of the line tangent to the given curve at the given point.

23 $y = x^2$; $(2, 4)$ **24** $y = 2x^2$; $(-1, 2)$

25 $y = -x^2$; $(0, 0)$ **26** $y = \sqrt{x}$; $(1, 1)$

27 $y = 3x - 4;\quad (1, -1)$ **28** $y = \sqrt{t-1};\quad (5, 2)$

29 $y = x^4;\quad (-2, 16)$ **30** $y = x^3 - x;\quad (0, 0)$

31 Find the equation of the line tangent to the parabola $y = x^2$ at the point (x_0, x_0^2).

32 Find all points $P(x_0, x_0^2)$ on the parabola $y = x^2$ such that the tangent line at P passes through the point $(0, -4)$.

□ **33** Prove that the line tangent to the parabola $y = x^2$ at $P(x_0, x_0^2)$ does not meet the parabola at any point except P.

2.3 DERIVATIVES OF RATIONAL FUNCTIONS

A term of the form

$$a_1x + a_0$$

where a_1, a_0 are real numbers, is called a *linear term* in x; if $a_1 \neq 0$, it is also called *polynomial of degree one* in x. A term of the form

$$a_2x^2 + a_1x + a_0, \qquad a_2 \neq 0$$

is called a *polynomial of degree two* in x, and, in general, a term of the form

$$a_nx^n + a_{n-1}x^{n-1} + \cdots + a_1x + a_0, \qquad a_n \neq 0$$

is called a *polynomial of degree n* in x.

A *rational term* in x is any term which is built up from the variable x and real numbers using the operations of addition, multiplication, subtraction, and division. For example every polynomial is a rational term and so are the terms

$$\frac{(3x^2 - 5)(x + 2)^3}{5x - 11}, \qquad \frac{(1 + 1/x)^9}{x^3 + 1/(2 - x)}.$$

A *linear function, polynomial function,* or *rational function* is a function which is given by a linear term, polynomial, or rational term, respectively. In this section we shall establish a set of rules which enable us to quickly differentiate any rational function. The rules will also be useful later on in differentiating other functions.

THEOREM 1

The derivative of a linear function is equal to the coefficient of x. That is,

$$\frac{d(bx + c)}{dx} = b, \qquad d(bx + c) = b\,dx.$$

PROOF Let $y = bx + c$, and let $\Delta x \neq 0$ be infinitesimal. Then

$$y + \Delta y = b(x + \Delta x) + c,$$

$$\Delta y = (b(x + \Delta x) + c) - (bx + c) = b\,\Delta x,$$

$$\frac{\Delta y}{\Delta x} = \frac{b\,\Delta x}{\Delta x} = b.$$

Therefore $$\frac{dy}{dx} = st(b) = b.$$

Multiplying through by dx, we obtain at once

$$dy = b\,dx.$$

If in Theorem 1 we put $b = 1, c = 0$, we see that the derivative of the identity function $f(x) = x$ is $f'(x) = 1$; i.e.,

$$\frac{dx}{dx} = 1, \qquad dx = dx.$$

On the other hand, if we put $b = 0$ in Theorem 1 then the term $bx + c$ is just the constant c, and we find that the derivative of the constant function $f(x) = c$ is $f'(x) = 0$; i.e.,

$$\frac{dc}{dx} = 0, \qquad dc = 0.$$

THEOREM 2 (Sum Rule)

Suppose u and v depend on the independent variable x. Then for any value of x where du/dx and dv/dx exist,

$$\frac{d(u+v)}{dx} = \frac{du}{dx} + \frac{dv}{dx}, \qquad d(u+v) = du + dv.$$

In other words, the derivative of the sum is the sum of the derivatives.

PROOF Let $y = u + v$, and let $\Delta x \neq 0$ be infinitesimal. Then

$$y + \Delta y = (u + \Delta u) + (v + \Delta v),$$

$$\Delta y = [(u + \Delta u) + (v + \Delta v)] - [u + v] = \Delta u + \Delta v,$$

$$\frac{\Delta y}{\Delta x} = \frac{\Delta u + \Delta v}{\Delta x} = \frac{\Delta u}{\Delta x} + \frac{\Delta v}{\Delta x}.$$

Taking standard parts,

$$st\left(\frac{\Delta y}{\Delta x}\right) = st\left(\frac{\Delta u}{\Delta x} + \frac{\Delta v}{\Delta x}\right) = st\left(\frac{\Delta u}{\Delta x}\right) + st\left(\frac{\Delta v}{\Delta x}\right).$$

Thus

$$\frac{dy}{dx} = \frac{du}{dx} + \frac{dv}{dx}.$$

By using the Sum Rule $n - 1$ times, we see that

$$\frac{d(u_1 + \cdots + u_n)}{dx} = \frac{du_1}{dx} + \cdots + \frac{du_n}{dx}, \quad \text{or} \quad d(u_1 + \cdots + u_n) = du_1 + \cdots + du_n.$$

THEOREM 3 (Constant Rule)

Suppose u depends on x, and c is a real number. Then for any value of x where du/dx exists,

$$\frac{d(cu)}{dx} = c\frac{du}{dx}, \qquad d(cu) = c\,du.$$

PROOF Let $y = cu$, and let $\Delta x \neq 0$ be infinitesimal. Then

$$y + \Delta y = c(u + \Delta u),$$
$$\Delta y = c(u + \Delta u) - cu = c\,\Delta u,$$
$$\frac{\Delta y}{\Delta x} = \frac{c\,\Delta u}{\Delta x} = c\frac{\Delta u}{\Delta x}.$$

Taking standard parts,

$$st\left(\frac{\Delta y}{\Delta x}\right) = st\left(c\frac{\Delta u}{\Delta x}\right) = c\,st\left(\frac{\Delta u}{\Delta x}\right)$$

whence
$$\frac{dy}{dx} = c\frac{du}{dx}.$$

The Constant Rule shows that in computing derivatives, a constant factor may be moved "outside" the derivative. It can only be used when c is a constant. For products of two functions of x, we have:

THEOREM 4 (Product Rule)

Suppose u and v depend on x. Then for any value of x where du/dx and dv/dx exist,

$$\frac{d(uv)}{dx} = u\frac{dv}{dx} + v\frac{du}{dx}, \qquad d(uv) = u\,dv + v\,du.$$

PROOF Let $y = uv$, and let $\Delta x \neq 0$ be infinitesimal.

$$y + \Delta y = (u + \Delta u)(v + \Delta v),$$
$$\Delta y = (u + \Delta u)(v + \Delta v) - uv = u\,\Delta v + v\,\Delta u + \Delta u\,\Delta v,$$
$$\frac{\Delta y}{\Delta x} = \frac{u\,\Delta v + v\,\Delta u + \Delta u\,\Delta v}{\Delta x} = u\frac{\Delta v}{\Delta x} + v\frac{\Delta u}{\Delta x} + \Delta u\frac{\Delta v}{\Delta x}.$$

Δu is infinitesimal by the Increment Theorem, whence

$$st\left(\frac{\Delta y}{\Delta x}\right) = st\left(u\frac{\Delta v}{\Delta x} + v\frac{\Delta u}{\Delta x} + \Delta u\frac{\Delta v}{\Delta x}\right)$$
$$= u \cdot st\left(\frac{\Delta v}{\Delta x}\right) + v \cdot st\left(\frac{\Delta u}{\Delta x}\right) + 0 \cdot st\left(\frac{\Delta v}{\Delta x}\right).$$

So
$$\frac{dy}{dx} = u\frac{dv}{dx} + v\frac{du}{dx}.$$

The Constant Rule is really the special case of the Product Rule where v is a constant function of x, $v = c$. To check this we let v be the constant c and see what the Product Rule gives us:

$$\frac{d(u \cdot c)}{dx} = u\frac{dc}{dx} + c\frac{du}{dx} = u \cdot 0 + c\frac{du}{dx} = c\frac{du}{dx}.$$

This is the Constant Rule.

The Product Rule can also be used to find the derivative of a power of u.

THEOREM 5 (Power Rule)

Let u depend on x and let n be a positive integer. For any value of x where du/dx exists,

$$\frac{d(u^n)}{dx} = nu^{n-1}\frac{du}{dx}, \qquad d(u^n) = nu^{n-1}\,du.$$

PROOF To see what is going on we first prove the Power Rule for $n = 1, 2, 3, 4$.

$n = 1$: We have $u^n = u$ and $u^0 = 1$, whence

$$\frac{d(u^n)}{dx} = \frac{du}{dx} = 1 \cdot u^0 \cdot \frac{du}{dx}.$$

$n = 2$: We use the Product Rule,

$$\frac{d(u^2)}{dx} = \frac{d(u \cdot u)}{dx} = u\frac{du}{dx} + u\frac{du}{dx} = 2 \cdot u^1 \cdot \frac{du}{dx}.$$

$n = 3$: We write $u^3 = u \cdot u^2$, use the Product Rule again, and then use the result for $n = 2$.

$$\frac{d(u^3)}{dx} = \frac{d(u \cdot u^2)}{dx} = u\frac{d(u^2)}{dx} + u^2\frac{du}{dx}$$

$$= u \cdot 2u\frac{du}{dx} + u^2\frac{du}{dx} = 3u^2\frac{du}{dx}.$$

$n = 4$: Using the Product Rule and then the result for $n = 3$,

$$\frac{d(u^4)}{dx} = \frac{d(u \cdot u^3)}{dx} = u\frac{d(u^3)}{dx} + u^3\frac{du}{dx}$$

$$= u \cdot 3u^2\frac{du}{dx} + u^3\frac{du}{dx} = 4u^3\frac{du}{dx}.$$

We can continue this process indefinitely and prove the theorem for every positive integer n. To see this, assume that we have proved the theorem for m. That is, assume that

(1) $$\frac{d(u^m)}{dx} = mu^{m-1}\frac{du}{dx}.$$

We then show that it is also true for $m + 1$. Using the Product Rule and the Equation 1,

$$\frac{d(u^{m+1})}{dx} = \frac{d(u \cdot u^m)}{dx} = u\frac{d(u^m)}{dx} + u^m\frac{du}{dx}$$

$$= u \cdot mu^{m-1}\frac{du}{dx} + u^m\frac{du}{dx} = (m + 1)u^m\frac{du}{dx}.$$

Thus $$\frac{d(u^{m+1})}{dx} = (m + 1)u^m\frac{du}{dx}.$$

This shows that the theorem holds for $m + 1$.

We have shown the theorem is true for 1, 2, 3, 4. Set $m = 4$; then the theorem holds for $m+1=5$. Set $m=5$; then it holds for $m+1=6$. And so on. Hence the theorem is true for all positive integers n.

In the proof of the Power Rule, we used the following principle:

PRINCIPLE OF INDUCTION

Suppose a statement $P(n)$ about an arbitrary integer n is true when $n=1$. Suppose further that for any positive integer m such that $P(m)$ is true, $P(m+1)$ is also true. Then the statement $P(n)$ is true of every positive integer n.

In the previous proof, $P(n)$ was the Power Rule,

$$\frac{d(u^n)}{dx} = nu^{n-1}\frac{du}{dx}.$$

The Principle of Induction can be made plausible in the following way. Let a positive integer n be given. Set $m=1$; since $P(1)$ is true, $P(2)$ is true. Now set $m=2$; since $P(2)$ is true, $P(3)$ is true. We continue reasoning in this way for n steps and conclude that $P(n)$ is true.

The Power Rule also holds for $n=0$ because when $u \neq 0, u^0 = 1$ and $d1/dx = 0$.

Using the Sum, Constant, and Power rules, we can compute the derivative of a polynomial function very easily. We have

$$\frac{d(x^n)}{dx} = nx^{n-1},$$

$$\frac{d(cx^n)}{dx} = cnx^{n-1},$$

and thus

$$\frac{d(a_nx^n + a_{n-1}x^{n-1} + \cdots + a_1x + a_0)}{dx} = a_n \cdot nx^{n-1} + a_{n-1}(n-1)x^{n-2} + \cdots + a_1.$$

EXAMPLE 1 $\dfrac{d(-3x^5)}{dx} = -3 \cdot 5x^4 = -15x^4.$

EXAMPLE 2 $\dfrac{d(6x^4 - 2x^3 + x - 1)}{dx} = 24x^3 - 6x^2 + 1.$

Two useful facts can be stated as corollaries.

COROLLARY 1

The derivative of a polynomial of degree $n > 0$ is a polynomial of degree $n - 1$. (A nonzero constant is counted as a polynomial of degree zero.)

COROLLARY 2

If u depends on x, then

$$\frac{d(u + c)}{dx} = \frac{du}{dx}$$

whenever du/dx exists. That is, adding a constant to a function does not change its derivative.

In Figure 2.3.1 we see that the effect of adding a constant is to move the curve up or down the y-axis without changing the slope.

For the last two rules in this section we need the formula for the derivative of $1/v$.

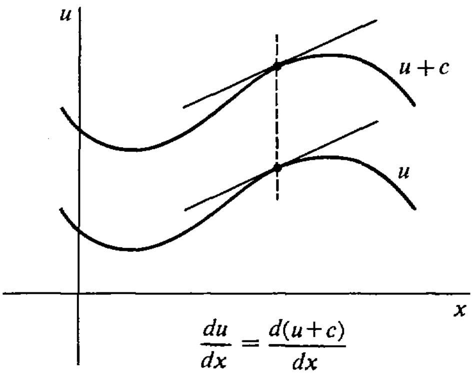

Figure 2.3.1

LEMMA

Suppose v depends on x. Then for any value of x where $v \neq 0$ and dv/dx exists,

$$\frac{d(1/v)}{dx} = -\frac{1}{v^2}\frac{dv}{dx}, \qquad d\left(\frac{1}{v}\right) = -\frac{1}{v^2}dv.$$

PROOF Let $y = 1/v$ and let $\Delta x \neq 0$ be infinitesimal.

$$y + \Delta y = \frac{1}{v + \Delta v},$$

$$\Delta y = \frac{1}{v + \Delta v} - \frac{1}{v},$$

$$\frac{\Delta y}{\Delta x} = \frac{1/(v + \Delta v) - 1/v}{\Delta x}$$

$$= \frac{v - (v + \Delta v)}{\Delta x v(v + \Delta v)}$$

$$= -\frac{1}{v(v + \Delta v)}\frac{\Delta v}{\Delta x}.$$

Taking standard parts,

$$st\left(\frac{\Delta y}{\Delta x}\right) = st\left(-\frac{1}{v(v+\Delta v)}\frac{\Delta v}{\Delta x}\right)$$
$$= st\left(-\frac{1}{v(v+\Delta v)}\right) st\left(\frac{\Delta v}{\Delta x}\right)$$
$$= -\frac{1}{v^2} st\left(\frac{\Delta v}{\Delta x}\right).$$

Therefore $$\frac{dy}{dx} = -\frac{1}{v^2}\frac{dv}{dx}.$$

THEOREM 6 (Quotient Rule)

Suppose u, v depend on x. Then for any value of x where du/dx, dv/dx exist and $v \neq 0$,

$$\frac{d(u/v)}{dx} = \frac{v\,du/dx - u\,dv/dx}{v^2}, \qquad d\left(\frac{u}{v}\right) = \frac{v\,du - u\,dv}{v^2}.$$

PROOF We combine the Product Rule and the formula for $d(1/v)$. Let $y = u/v$. We write y in the form

$$y = \frac{1}{v}\cdot u.$$

Then $$dy = d\left(\frac{1}{v}u\right) = \frac{1}{v}du + u\,d\left(\frac{1}{v}\right)$$
$$= \frac{1}{v}du + u(-v^{-2})\,dv$$
$$= \frac{v\,du - u\,dv}{v^2}.$$

THEOREM 7 (Power Rule for Negative Exponents)

Suppose u depends on x and n is a negative integer. Then for any value of x where du/dx exists and $u \neq 0$, $d(u^n)/dx$ *exists and*

$$\frac{d(u^n)}{dx} = nu^{n-1}\frac{du}{dx}, \qquad d(u^n) = nu^{n-1}\,du.$$

PROOF Since n is negative, $n = -m$ where m is positive. Let $y = u^n = u^{-m}$. Then $y = 1/u^m$. By the Lemma and the Power Rule,

$$\frac{dy}{dx} = -\frac{1}{(u^m)^2}\cdot\frac{d(u^m)}{dx}$$
$$= -\frac{1}{u^{2m}}\cdot mu^{m-1}\frac{du}{dx}$$

$$= (-m)u^{-2m}u^{m-1}\frac{du}{dx}$$

$$= (-m)u^{-m-1}\frac{du}{dx} = nu^{n-1}\frac{du}{dx}.$$

The Quotient Rule together with the Constant, Sum, Product, and Power Rules make it easy to differentiate any rational function.

EXAMPLE 3 Find dy when

$$y = \frac{1}{x^2 - 3x + 1}.$$

Introduce the new variable u with the equation

$$u = x^2 - 3x + 1.$$

Then $y = 1/u$, and $du = (2x - 3)\,dx$, so

$$dy = -\frac{1}{u^2}du = \frac{-(2x-3)}{(x^2 - 3x + 1)^2}dx.$$

EXAMPLE 4 Let $y = \dfrac{(x^4 - 2)^3}{5x - 1}$ and find dy.

Let $\quad u = (x^4 - 2)^3, \qquad v = 5x - 1.$

Then $\quad y = \dfrac{u}{v}, \qquad dy = \dfrac{v\,du - u\,dv}{v^2}.$

Also, $\quad du = 3 \cdot (x^4 - 2)^2 \cdot 4x^3\,dx = 12(x^4 - 2)^2 \cdot x^3\,dx,$

$dv = 5\,dx.$

Therefore

$$dy = \frac{(5x - 1)12(x^4 - 2)^2x^3\,dx - (x^4 - 2)^3 5\,dx}{(5x - 1)^2}$$

$$= \frac{(x^4 - 2)^2[12(5x - 1)x^3 - 5(x^4 - 2)]}{(5x - 1)^2}dx.$$

EXAMPLE 5 Let $y = 1/x^3 + 3/x^2 + 4/x + 5$.

Then

$$dy = \left(-\frac{3}{x^4} - \frac{6}{x^3} - \frac{4}{x^2}\right)dx.$$

EXAMPLE 6 Find dy where

$$y = \left(\frac{1}{x^2 + x} + 1\right)^2.$$

This problem can be worked by means of a double substitution. Let

$$u = x^2 + x, \qquad v = \frac{1}{u} + 1.$$

Then $$y = v^2.$$

We find dy, dv, and du,

$$dy = 2v\,dv,$$
$$dv = -u^{-2}\,du,$$
$$du = (2x + 1)\,dx.$$

Substituting, we get dy in terms of x and dx,

$$\begin{aligned} dy &= 2v(-u^{-2}\,du) \\ &= -2vu^{-2}(2x + 1)\,dx \\ &= -2\left(\frac{1}{u} + 1\right)u^{-2}(2x + 1)\,dx \\ &= -2\left(\frac{1}{x^2 + x} + 1\right)(x^2 + x)^{-2}(2x + 1)\,dx. \end{aligned}$$

EXAMPLE 7 Assume that u and v depend on x. Given $y = (uv)^{-2}$, find dy/dx in terms of du/dx and dv/dx.

Let $s = uv$, whence $y = s^{-2}$. We have

$$dy = -2s^{-3}\,ds,$$
$$ds = u\,dv + v\,du.$$

Substituting, $$dy = -2(uv)^{-3}(u\,dv + v\,du),$$

and $$\frac{dy}{dx} = -2(uv)^{-3}\left(u\frac{dv}{dx} + v\frac{du}{dx}\right).$$

The six rules for differentiation which we have proved in this section are so useful that they should be memorized. We list them all together.

Table 2.3.1 Rules for Differentiation

(1)	$\dfrac{d(bx + c)}{dx} = b.$	$d(bx + c) = b\,dx.$
(2)	$\dfrac{d(u + v)}{dx} = \dfrac{du}{dx} + \dfrac{dv}{dx}.$	$d(u + v) = du + dv.$
(3)	$\dfrac{d(cu)}{dx} = c\dfrac{du}{dx}.$	$d(cu) = c\,du.$
(4)	$\dfrac{d(uv)}{dx} = u\dfrac{dv}{dx} + v\dfrac{du}{dx}.$	$d(uv) = u\,dv + v\,du.$
(5)	$\dfrac{d(u^n)}{dx} = nu^{n-1}\dfrac{du}{dx}.$	$d(u^n) = nu^{n-1}\,du$ (n is any integer).
(6)	$\dfrac{d(u/v)}{dx} = \dfrac{v\,du/dx - u\,dv/dx}{v^2}.$	$d(u/v) = \dfrac{v\,du - u\,dv}{v^2}.$

An easy way to remember the way the signs are in the Quotient Rule 6 is to put $u = 1$ and use the Power Rule 5 with $n = -1$,

$$d(1/v) = d(v^{-1}) = -1 \cdot v^{-2}\, dv = \frac{-1\, dv}{v^2}.$$

PROBLEMS FOR SECTION 2.3

In Problems 1–42 below, find the derivative.

1 $f(x) = 3x^2 + 5x - 4$

2 $s = \frac{1}{3}t^3 + \frac{1}{2}t^2 + t$

3 $y = (x + 8)^5$

4 $z = (2 + 3x)^4$

5 $f(t) = (4 - t)^3$

6 $g(x) = 3(2 - 5x)^6$

7 $y = (x^2 + 5)^3$

8 $u = (6 + 2x^2)^3$

9 $u = (6 - 2x^2)^3$

10 $w = (1 + 4x^3)^{-2}$

11 $w = (1 - 4x^3)^{-2}$

12 $y = 1 + x^{-1} + x^{-2} + x^{-3}$

13 $f(x) = 5(x + 1 - 1/x)$

14 $u = (x^2 + 3x + 1)^4$

15 $v = 4(2x^2 - x + 3)^{-2}$

16 $y = -(2x + 3 + 4x^{-1})^{-1}$

17 $y = \dfrac{1}{1 + 1/t}$

18 $y = \dfrac{1}{2x^2 + 1}$

19 $s = \dfrac{-3}{4t^2 - 2t + 1}$

20 $s = (2t + 1)(3t - 2)$

21 $h(x) = \frac{1}{2}(x^2 + 1)(5 - 2x)$

22 $y = (2x^3 + 4)(x^2 - 3x + 1)$

23 $v = (3t^2 + 1)(2t - 4)^3$

24 $z = (-2x + 4 + 3x^{-1})(x + 1 - 5x^{-1})$

25 $y = \dfrac{x + 1}{x - 1}$

26 $w = \dfrac{2 - 3x}{1 + 2x}$

27 $y = \dfrac{x^2 - 1}{x^2 + 1}$

28 $u = \dfrac{x}{x^2 + 1}$

29 $x = \dfrac{(s - 1)(s - 2)}{s - 3}$

30 $y = \dfrac{t}{1 + 1/t}$

31 $y = \dfrac{2x^{-1} - x^{-2}}{3x^{-1} - 4x^{-2}}$

32 $y = 4x - 5$

33 $y = 6$

34 $y = 2x(3x - 1)(4 - 2x)$

35 $y = 3(x^2 + 1)(2x^2 - 1)(2x + 3)$

36 $y = (4x + 3)^{-1} + (x - 4)^{-2}$

37 $z = \dfrac{1}{(2x + 1)(x - 3)}$

38 $y = (x^2 + 1)^{-1}(3x - 1)^{-2}$

39 $y = [(2x + 1)^{-1} + 3]^{-1}$

40 $s = [(t^2 + 1)^3 + t]^{-1}$

41 $y = (2x + 1)^3(x^2 + 1)^2$

42 $y = \left(\dfrac{2}{x - 1} - x^{-3}\right)^4$

In Problems 43–48, assume u and v depend on x and find dy/dx in terms of du/dx and dv/dx.

43 $y = u - v$

44 $y = u^2 v$

45 $y = 4u + v^2$

46 $y = 1/(u + v)$

47 $y = 1/uv$

48 $y = (u + v)(2u - v)$

49 Find the line tangent to the curve $y = 1 + x + x^2 + x^3$ at the point $(1, 4)$.