$$\frac{d(e^x)}{dx} = e^x.$$

In other words, e is the real number such that the constant

$$st\left[\frac{e^{\Delta x} - 1}{\Delta x}\right] = 1$$

(where Δx is a nonzero infinitesimal). It will be shown in Section 8.3 that there is such a number e and that e has the approximate value

$$e \sim 2.71828.$$

The function $y = e^x$ is called the *exponential function.* e^x is always positive and follows the rules

$$e^{a+b} = e^a \cdot e^b, \qquad e^{a \cdot b} = (e^a)^b, \qquad e^0 = 1.$$

Figure 2.5.7 shows the graph of $y = e^x$.

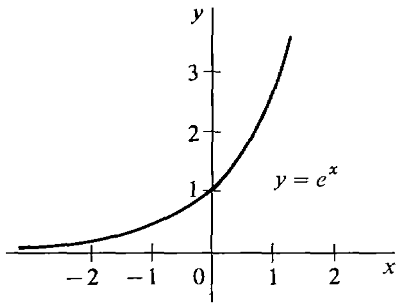

Figure 2.5.7

EXAMPLE 3 Find the derivative of $y = x^2e^x$. By the Product Rule,

$$\frac{dy}{dx} = x^2\frac{d(e^x)}{dx} + e^x\frac{d(x^2)}{dx} = x^2e^x + 2xe^x.$$

3 THE NATURAL LOGARITHM

The inverse of the exponential function $x = e^y$ is the *natural logarithm function,* written

$$y = \ln x.$$

Verbally, $\ln x$ is the number y such that $e^y = x$. Since $y = \ln x$ is the inverse function of $x = e^y$, we have

$$e^{\ln a} = a, \qquad \ln(e^a) = a.$$

The simplest values of $y = \ln x$ are

$$\ln(1/e) = -1, \qquad \ln(1) = 0, \qquad \ln e = 1.$$

Figure 2.5.8 shows the graph of $y = \ln x$. It is defined only for $x > 0$.

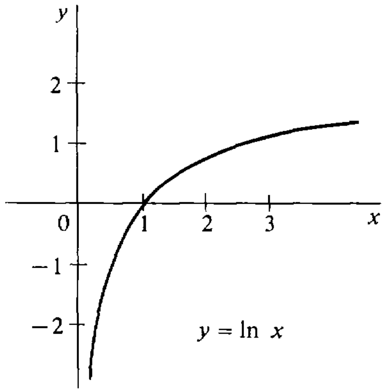

Figure 2.5.8

The most important rules for logarithms are

$$\ln(ab) = \ln a + \ln b,$$
$$\ln(a^b) = b \cdot \ln a.$$

The natural logarithm function is important in calculus because its derivative is simply $1/x$,

$$\frac{d(\ln x)}{dx} = \frac{1}{x}, \qquad (x > 0).$$

This can be derived from the Inverse Function Rule.

If $\quad y = \ln x,$

then $\quad x = e^y,$

$$\frac{dx}{dy} = e^y,$$

$$\frac{dy}{dx} = \frac{1}{dx/dy} = \frac{1}{e^y} = \frac{1}{x}.$$

The natural logarithm is also called the logarithm to the base e and is sometimes written $\log_e x$. Logarithms to other bases are discussed in Chapter 8.

EXAMPLE 4 Differentiate $y = \dfrac{1}{\ln x}$.

$$\frac{dy}{dx} = \frac{-1}{(\ln x)^2}\frac{d(\ln x)}{dx} = -\frac{1}{x(\ln x)^2}.$$

4 SUMMARY

Here is a list of the new derivatives given in this section.

$$\frac{d(\sin x)}{dx} = \cos x.$$

$$\frac{d(\cos x)}{dx} = -\sin x.$$

$$\frac{d(e^x)}{dx} = e^x.$$

$$\frac{d(\ln x)}{dx} = \frac{1}{x} \quad (x > 0).$$

Tables of values for $\sin x$, $\cos x$, e^x, and $\ln x$ can be found at the end of the book.

PROBLEMS FOR SECTION 2.5

In Problems 1–20, find the derivative.

1 $y = \cos^2 \theta$ **2** $s = \tan^2 t$

3 $y = 2 \sin x + 3 \cos x$ **4** $y = \sin x \bullet \cos x$

5 $w = \dfrac{1}{\cos z}$ **6** $w = \dfrac{1}{\sin z}$

7 $y = \sin^n \theta$ **8** $y = \tan^n \theta$

9 $s = t \sin t$ **10** $s = \dfrac{\cos t}{t - 1}$

11 $y = xe^x$ **12** $y = 1/(1 + e^x)$

13 $y = (\ln x)^2$ **14** $y = x \ln x$

15 $y = e^x \bullet \ln x$ **16** $y = e^x \bullet \sin x$

17 $u = \sqrt{v}(1 - e^v)$ **18** $u = (1 + e^v)(1 - e^v)$

19 $y = x^n \ln x$ **20** $y = (\ln x)^n$

In Problems 21–24, find the equation of the tangent line at the given point.

21 $y = \sin x$ at $(\pi/6, \frac{1}{2})$ **22** $y = \cos x$ at $(\pi/4, \sqrt{2}/2)$

23 $y = x - \ln x$ at $(e, e - 1)$ **24** $y = e^{-x}$ at $(0, 1)$

2.6 CHAIN RULE

The Chain Rule is more general than the Inverse Function Rule and deals with the case where x and y are both functions of a third variable t.

Suppose $$x = f(t), \qquad y = G(x).$$

Thus x depends on t, and y depends on x. But y is also a function of t,

$$y = g(t),$$

where g is defined by the rule

$$g(t) = G(f(t)).$$

The function g is sometimes called the *composition* of G and f (sometimes written $g = G \circ f$).