Ch. 9: Momentum

$\vec{p} = m\vec{v}$ (momentum)

$$\frac{d\vec{p}}{dt} = m\frac{d\vec{v}}{dt} = m\vec{a} = \sum \vec{F}$$

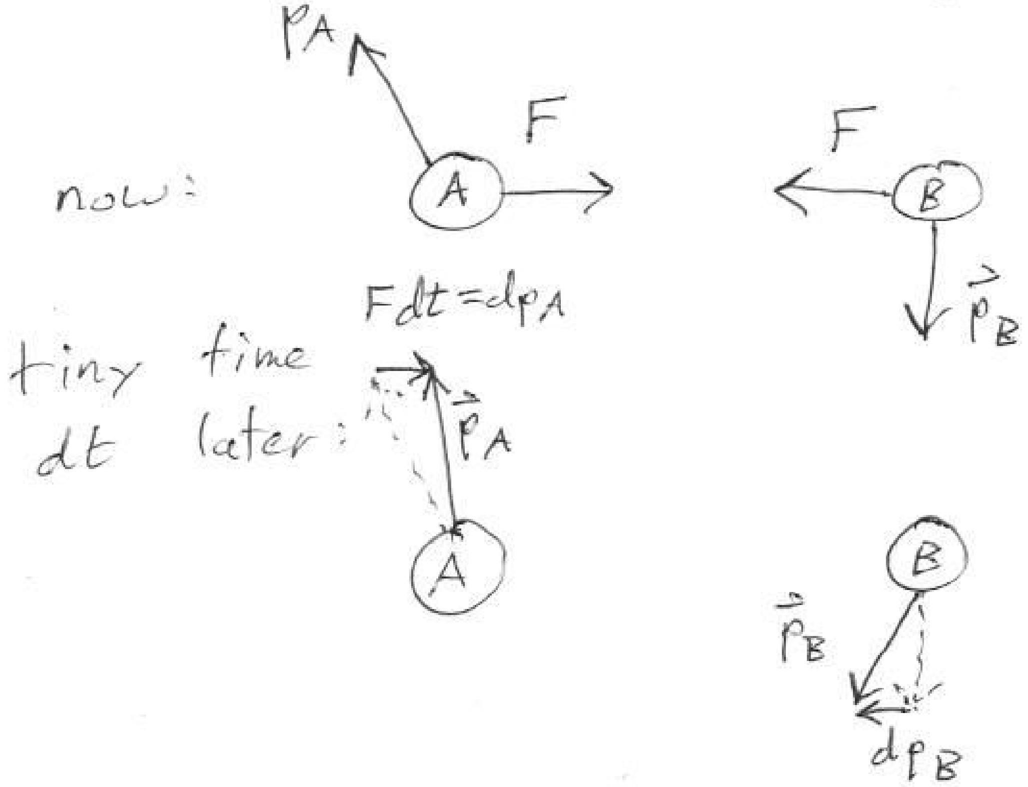
if m constant

sum of forces acting on mass m.

Newton's 1st Law: $\sum \vec{F} = \vec{0} \Rightarrow \vec{p}$ constant

2nd Law: $d\vec{p}/dt = \sum \vec{F}$

3rd Law: For every change in momentum, there is an equal & opposite change in momentum.



When m is not constant,

$\sum \vec{F} \neq m\vec{a}$, but $\sum \vec{F} = \frac{d\vec{p}}{dt}$ holds.

$$\sum \vec{F} = \frac{d(m\vec{v})}{dt} = \frac{dm}{dt}\vec{v} + m\frac{d\vec{v}}{dt}$$

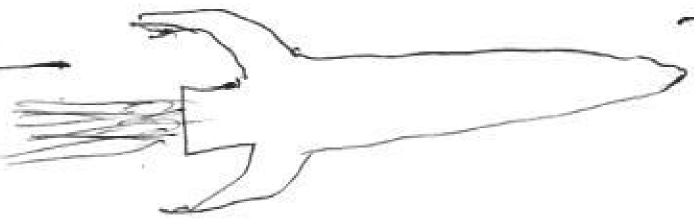
More extreme example: light has no mass, but it does have momentum.

Another way to state Newton's 3rd Law: momentum is conserved.

Consider a rocket in deep space (no external forces acting on it).

Rocket is releasing exhaust gas at a constant rate b of mass per time.

$$\longleftarrow bu = \frac{dp_{ex}}{dt}$$



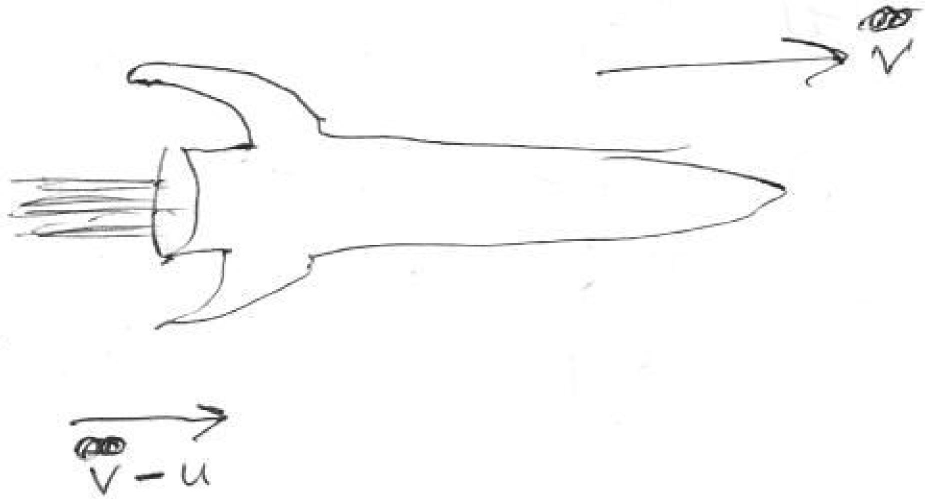
$$\longrightarrow bu = \frac{dp_{ro}}{dt}$$

relative to rocket

The exhaust gas is leaving the rocket at constant speed u relative to rocket.

relative to rocket

The total momentum of all the exhaust gas is changing at at rate bu to the left, relative to the rocket a moment ago. So the rocket's momentum changes at a rate of bu to the right, relative to where the rocket ~~[illegible]~~ a moment ago. An external, stationary observer will see the rocket travelling at some velocity v to the right & the exhaust ~~travelling~~ ~~at~~ exiting the rocket with velocity $v-u$.



$$\frac{dp_{Ex}}{dt} = b(v-u) \qquad \frac{dp_{Ro}}{dt} = b(u-v)$$

Let m = rocket mass. $\quad \frac{dm}{dt} = -b.$

$$\frac{d(mv)}{dt} = b(u-v) \qquad m = \underbrace{m_0}_{\text{mass when } t=0} - bt$$

$$\frac{dm}{dt}v + m\frac{dv}{dt} = b(u-v)$$

$$-bv + m\frac{dv}{dt} = bu - bv$$

$$m\frac{dv}{dt} = bu$$

$$(m_0 - bt)\frac{dv}{dt} = bu$$

$$dv = \frac{bu\,dt}{m_0 - bt}$$

For a simulation: dt small

$v = v + dv$

incrementing v by dv

$$\Delta v = v - v_0 = \int_{v_0}^{v} dv = \int_0^t \frac{bu\,dt}{m_0 - bt}$$

substitute $w = m_0 - bt$ $\qquad t = (m_0 - w)/b$

$dt = -dw/b$

$$\Delta v = \int_{m_0}^{m_0 - bt} \frac{-bu\, dw/b}{w} = \int_{m_0}^{m_0-bt} -u \frac{dw}{w}$$

$$\Delta v = -u \ln|w| \Big|_{m_0}^{m_0-bt} = -u\left[\ln(m_0 - bt) - \ln m_0\right]$$

$$= u\left[\ln m_0 - \ln(m_0 - bt)\right] = u \ln\frac{m_0}{m_0 - bt}$$

(Book: Section 9-10.)

Simulation: Start: $m_0,\ v_0,\ x_0,\ t_0 = 0,\ \overbrace{b,\ u}^{\text{constant}}$

$m = m_0,\ v = v_0,\ x = x_0,\ t = t_0,\ dt.$

Iterate: increment m, v, x, t:

$t = t + dt$

$m = m - b\,dt$

$dv = (b\,u\,dt/m) + (\underbrace{a_G}_{\text{acceleration from gravity}}\,dt)$

$v = v + dv$

$dx = v\,dt$

$x = x + dx$

Collisions / Colissions / Collissions

Two extremes:

Completely elastic:

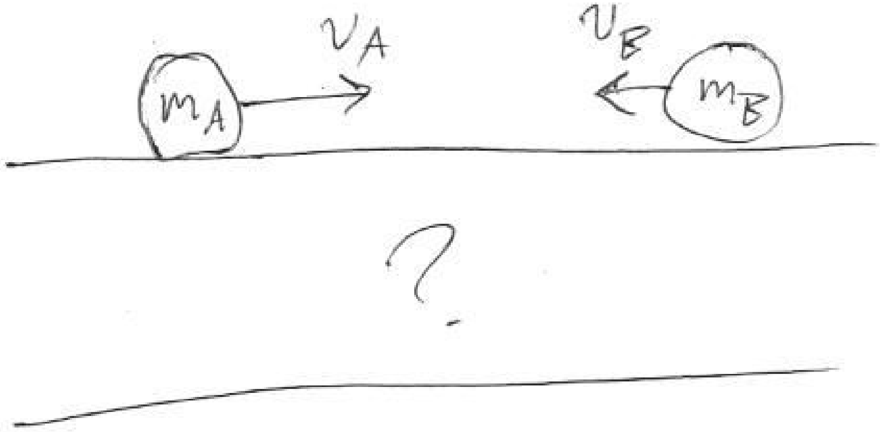
total kinetic energy conserved

Completely inelastic:

objects stick together after collisions

Reality is usually somewhere in between.

But in all collissions, momentum is conserved.



Before

After

E.g. $m_A = 5.0\,kg$ $m_B = 6.0\,kg$

Before: $v_{Ax} = 4.0\,m/s$

$\longrightarrow x$ $v_{Bx} = -1.0\,m/s$

Total momentum $p_x = m_A v_{Ax} + m_B v_{Bx}$

is same before & after

Let v_{Ax}^* & v_{Bx}^* be the velocities

after the collission:

$$m_A v_{Ax} + m_B v_{Bx} = m_A v_{Ax}^* + m_B v_{Bx}^*$$

Completely elastic case:

$$K_{total} = \frac{1}{2} m_A v_A^2 + \frac{1}{2} m_B v_B^2 = \frac{1}{2} m_A \left(v_{Ax}^*\right)^2 + \frac{1}{2} m_B \left(v_{Bx}^*\right)^2$$

Completely inelastic case:

$$v_A^* = v_B^*.$$

$p_x = 14.0\ \mathrm{kg\ m/s}$ $(\mathrm{kg\cdot m/s} = \mathrm{N\cdot s})$

$$\boxed{p_x = \overbrace{m_A}^{5.0\,\mathrm{kg}} v^*_{Ax} + \overbrace{m_B}^{6.0\,\mathrm{kg}} v^*_{Bx}}$$

$$\underbrace{K_{total}}_{43\,\mathrm{J}} = \underbrace{\tfrac{1}{2}(5.0\,\mathrm{kg})(4.0\,\mathrm{m/s})^2}_{40\,\underbrace{\mathrm{kg\,m^2/s^2}}_{\mathrm{J}}} + \underbrace{\tfrac{1}{2}(6.0\,\mathrm{kg})(-1.0\,\mathrm{m/s})^2}_{3.0\,\underbrace{\mathrm{kg\,m^2/s^2}}_{\mathrm{J}}}$$

$\mathrm{J} = \mathrm{N\cdot m}$

$$\boxed{43\,\mathrm{J} = K_{total} = \tfrac{1}{2}\underbrace{m_A}_{5.0\,\mathrm{kg}}\left(v^*_{Ax}\right)^2 + \tfrac{1}{2}\underbrace{m_B}_{6.0\,\mathrm{kg}}\left(v^*_{Bx}\right)^2}$$

$$v^*_{Bx} = \left(p_x - m_A v^*_{Ax}\right) / m_B$$

$$K_{tot} = \tfrac{1}{2} m_A \left(v^*_{Ax}\right)^2 + \tfrac{1}{2} m_B \left(\left(p_x - m_A v^*_{Ax}\right)/m_B\right)^2$$

$$K_{tot} = \tfrac{1}{2} m_A \left(v^*_{Ax}\right)^2 + \tfrac{1}{2} m_B \frac{\left(p_x^2 - 2 p_x m_A v^*_{Ax} + m_A^2 \left(v^*_{Ax}\right)^2\right)}{m_B^2}$$

$$K_{tot} = \tfrac{1}{2}\left(m_A + \frac{m_A^2}{m_B}\right)\left(v^*_{Ax}\right)^2 - \frac{m_A}{m_B} p_x v^*_{Ax} + \frac{p_x^2}{2 m_B}$$

$$2 m_B K_{tot} = m_A (m_B + m_A)\left(v^*_{Ax}\right)^2 - 2 m_A p_x v^*_{Ax} + p_x^2$$

$$0 = \underbrace{m_A (m_A + m_B)}_{a} (v_{Ax}^*)^2 \underbrace{-2 m_A p_x}_{b} v_{Ax}^* + \underbrace{p_x^2 - 2 m_B K_{tot}}_{c}$$

$$v_{Ax}^* = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

Plug solution into $v_{Bx}^* = \dfrac{p_x - m_A v_{Ax}^*}{m_B}$

$\pm \Rightarrow$ 2 solutions.

One solutions corresponds to

$v_{Ax}^* = v_{Ax}$ & $v_{Bx}^* = v_{Bx}$:

the objects pass through each other.

The other solution is probably what you want.

$a = 55\,\mathrm{kg}^2$ $\quad b = -140\,\mathrm{kg}^2\,\mathrm{m/s}$

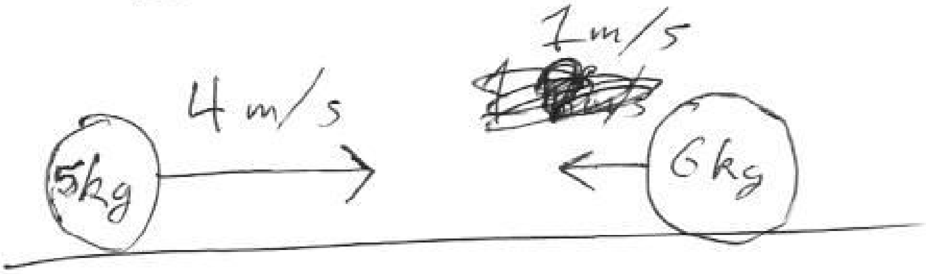
$c = -320\,\mathrm{kg}^2\,\mathrm{m}^2/\mathrm{s}^2$

$$v^*_{Ax} = -1.45\text{ m/s}, \quad \underbrace{4.00\text{ m/s}}_{v_{Ax}}$$

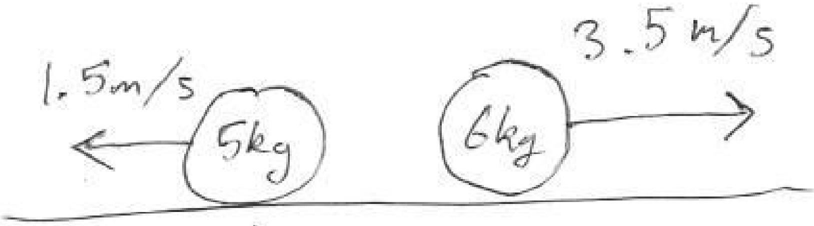
$$v^*_{Bx} = 3.54\text{ m/s}, \quad \underbrace{-1.00\text{ m/s}}_{v_{Bx}}$$

Solution is:

$$\left.\begin{aligned} v^*_{Ax} &= -1.5\text{ m/s} \\ v^*_{Bx} &= 3.5\text{ m/s} \end{aligned}\right] \text{ Completely elastic case}$$



before



after

Read Chapter 9

E.g. I didn't center of mass here.