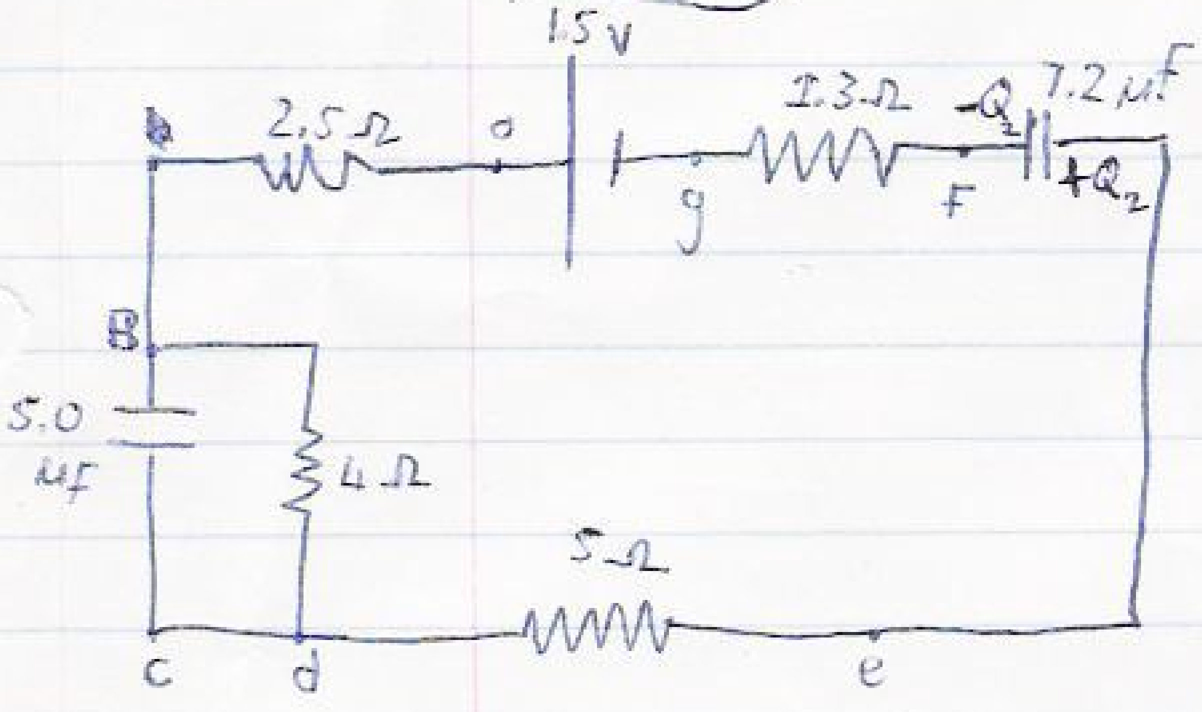
Notes.



- If the capacitors are fully charged how much energy is stored in each?

- $U = \frac{1}{2} QV = \frac{1}{2} \frac{Q^2}{C} = \frac{1}{2} CV^2$

*. $I = 0$, fully charged.
∴ No voltage drop anywhere

- $1.5\,V$ = Voltage drop along abcdefg = 0
$= 0(2.5\,\Omega) + V_{bc} + 0(5\,\Omega) + V_{ef} + 0(1.3\,\Omega)$
$1.5\,V = \underbrace{V_{bc}}_{0} + V_{ef}$

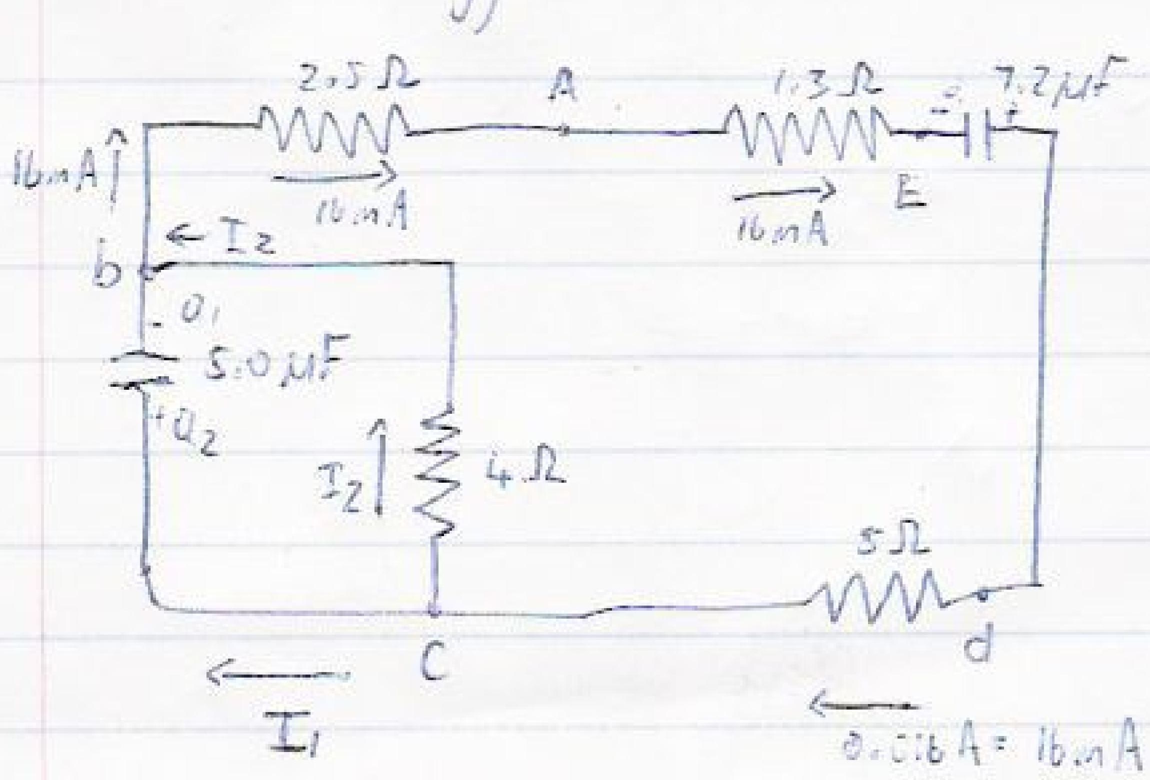
0 voltage drop along bcdh: $V_{bc} + 0(4\,\Omega) = V_{bc}$

- $1.5\,V = V_{ef} \Rightarrow U = \frac{1}{2}(7.2\,\mu F)(1.5\,V)^2$
$= \frac{1}{2} \cdot (7.2 \times 10^{-6}) \frac{\text{coulomb}}{\text{volt}} \; 1.5^2 \text{ volt}^2$

7.2 μF capacitor $\therefore U = 8.1 \times 10^{-6}$ coulomb.volt / Joules (J)

- Now suppose we remove the battery (and close the circuit) and very soon after measure the current across the 5 Ω resistor to be 0.016 A

- How much energy is then stored in each capacitor.



see circuit on next page

and on previous page

Loop cbc: $0 = \underbrace{V_{cb}}_{\substack{V_c - V_b \\ >0}} + \underbrace{V_{bc}}_{\substack{V_b - V_c \\ >0}} = \frac{Q_1}{5.0\,\mu F} - I_2(4\Omega)$

Loop dcfbaed: $0 = (16mA)(5\Omega) + I_2(4\Omega) + (16mA)(2.5\Omega) + (16mA)(1.3\Omega) - \frac{Q_2}{7.2\mu F}$

Junction c: $I_1 + I_2 = 16mA$

$\frac{dQ_1}{dt} = I_1$ $\qquad$ $\frac{dQ_2}{dt} = -16mA$

at a certain moment in time, but not at all times.

~~Conservation of charge: $Q_1 + Q_2 = \underbrace{(7.2\mu F)(1.5V)}_{constant}$~~

~~$\frac{dQ_1}{dt} + \frac{dQ_2}{dt} = 0$~~

~~Now I can show solve for Q_1 and Q_2~~

~~Then $U_1 = \frac{1}{2}\frac{Q_1^2}{5.0\mu F}$ and $U_2 = \frac{1}{2}\frac{Q_2^2}{7.2\mu F}$~~

~~$\frac{dQ_1}{dt} + \underbrace{\frac{dQ_2}{dt}}_{-16mA} = 0$~~

~~$\frac{dQ_1}{dt} = 16mA$~~

~~$I_1 = \underbrace{dQ_1/dt}_{16mA}$~~

~~$\underbrace{16mA}_{I_1} + I_2 = 16mA$~~

$I_2 = 0$

~~$0 = \frac{Q_1}{5.0\mu F}$~~

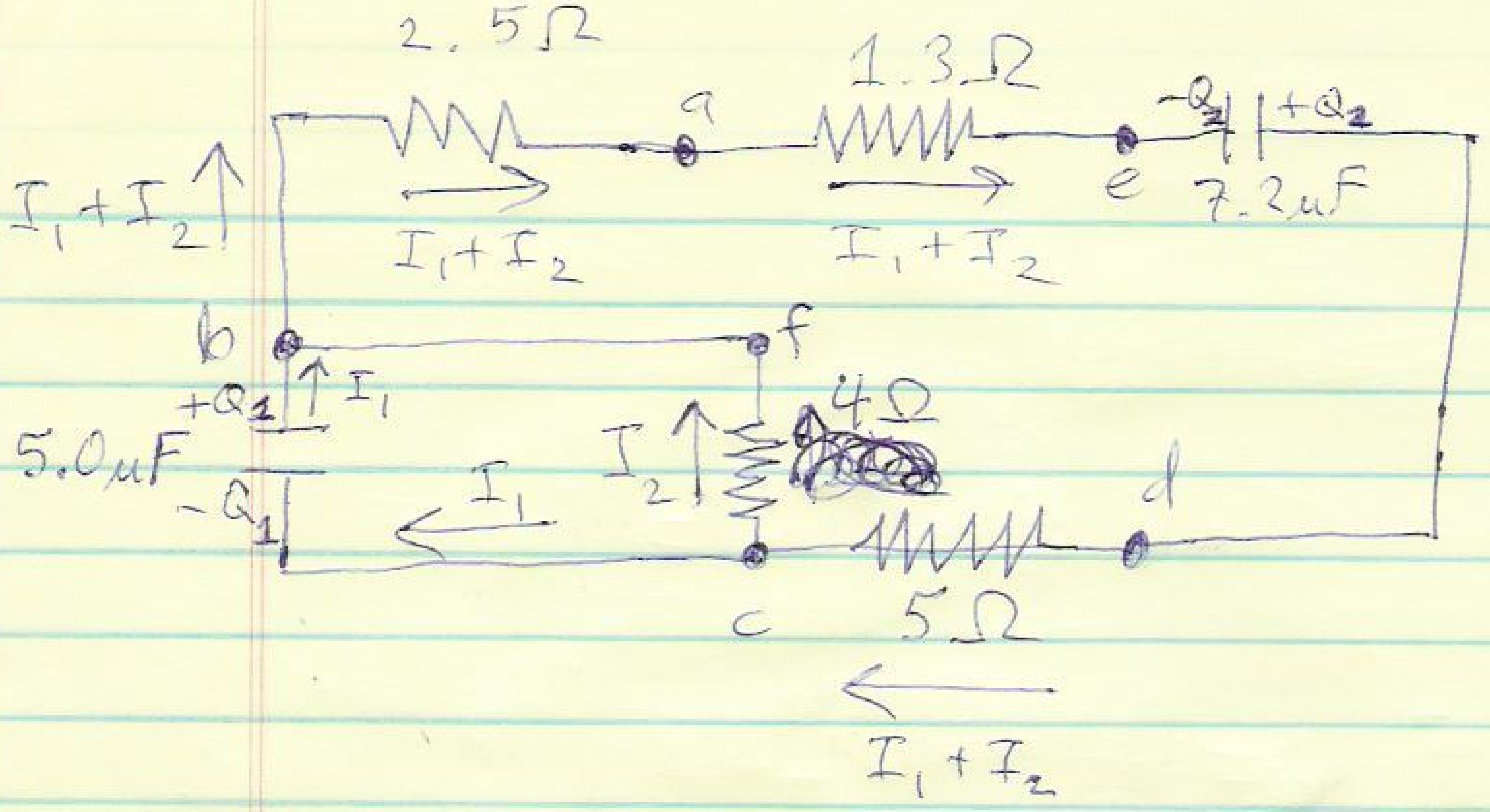
At all times:

$$0 = (I_1 + I_2)(5\Omega) + I_2(4\Omega) + (I_1 + I_2)(2.5\Omega) + (I_1 + I_2)(1.3\Omega) - \frac{Q_2}{7.2\mu F}$$

$dQ_1/dt = I_1$ &

$$\frac{dQ_2}{dt} = -(I_1 + I_2)$$

$$\Rightarrow 0 = \left(-\frac{dQ_2}{dt}\right)(5\Omega) + I_2(4\Omega) + \left(-\frac{dQ_2}{dt}\right)(2.5\Omega) + \left(-\frac{dQ_2}{dt}\right)(1.3\Omega) - \frac{Q_2}{7.2\mu F}$$



Solving equation of Loop cbc: $I_2 = \dfrac{Q_1}{(4\,\Omega)(5.0\,\mu F)} = \dfrac{Q_1}{\tau_1}$

where $\tau_1 = (4\,\Omega)(5.0\,\mu F) = 2 \cdot 10^{-5}$ seconds

$$\Rightarrow \frac{dQ_1}{dt} = \underbrace{I_1 = -\frac{dQ_2}{dt} - I_2}_{\text{because } \frac{dQ_2}{dt} = -(I_1+I_2)} = -\frac{dQ_2}{dt} - \frac{Q_1}{\tau_1}$$

Also
$$0 = \left(-\frac{dQ_2}{dt}\right)(5\,\Omega) + \frac{Q_1}{\tau_1}(4\,\Omega) + \left(-\frac{dQ_2}{dt}\right)(2.5\,\Omega) + \left(-\frac{dQ_2}{dt}\right)(1.3\,\Omega) - \frac{Q_2}{7.2\,\mu F} = \left\{ \frac{Q_1(4\,\Omega)}{\tau_1} - \frac{Q_2}{7.2\,\mu F} - \frac{dQ_2}{dt}(8.8\,\Omega) \right.$$

$$\Rightarrow \frac{dQ_2}{dt} = \frac{Q_1\,(4\,\Omega)}{\tau_1\,(8.8\,\Omega)} - \frac{Q_2}{(7.2\,\mu F)(8.8\,\Omega)}$$

$$\Rightarrow \frac{dQ_1}{dt} = -\frac{dQ_2}{dt} - \frac{Q_1}{\tau_1} = -\frac{Q_1\,(4)}{8.8\,\tau_1} + \frac{Q_2}{\underset{\uparrow}{\cancel{\tau_{21}}}} - \frac{Q_1}{\tau_1}$$

$(7.2\,\mu F)(8.8\,\Omega)$

$$\Rightarrow \boxed{\frac{dQ_1}{dt} = \frac{Q_2}{\tau_{21}} - \frac{Q_1}{\tau_{11}}} \quad (1)$$

where
$$\begin{cases} \tau_{21} = (7.2\,\mu F)(8.8\,\Omega) \ \& \\ \dfrac{1}{\tau_{11}} = \dfrac{4}{8.8\,\tau_1} + \dfrac{1}{\tau_1} \end{cases}$$

Also
$$\boxed{\frac{dQ_2}{dt} = \frac{Q_1}{\tau_{12}} - \frac{Q_2}{\tau_{22}}} \quad (2)$$

where $\dfrac{1}{\tau_{12}} = \dfrac{4}{8.8\,\tau_1}$ & $\tau_{22} = \overbrace{(7.2\,\mu F)(8.8\,\Omega)}^{=\tau_{21}}$.

Initially, at time $t=0$, $\begin{cases} Q_2 = (7.2\,\mu F)(1.5\,V) \\ Q_1 = 0 = Q_{10} \end{cases}$ (3)

Let $Q_{20} = (7.2\,\mu F)(1.5\,V) = \boxed{1.08 \cdot 10^{-5}\,C}$

We can solve (1), (2), (3) to find Q_1 & Q_2 as functions of time.

Then, we differentiate to get $\dfrac{dQ_2}{dt}$ as a function of time.

To find e^{At}, we diagonalize A:

$A = B^{-1} D B$ where $D = \begin{bmatrix} d_1 & 0 \\ 0 & d_2 \end{bmatrix}$ is diagonal.

Then $e^{At} = B^{-1} e^{Dt} B = B^{-1} \begin{bmatrix} e^{d_1 t} & 0 \\ 0 & e^{d_2 t} \end{bmatrix} B$

You can find B & D by hand (it involves determinants, the quadratic formula, and some more systems of equations) or you can compute it numerically using a computer program or calculator program; (Finding B^{-1} can then be ~~done~~ found by hand or by computer.)

$$\left.\begin{aligned} \tau_{12}^{-1} &= 2.272727\ldots \cdot 10^4/s \\ \tau_{11}^{-1} &= 7.272727\ldots \cdot 10^4/s \\ \tau_{21}^{-1} = \tau_{22}^{-1} &= 1.578282\ldots \cdot 10^4/s \end{aligned}\right\} \Rightarrow A = \begin{bmatrix} -7.2727\ldots & 1.578\ldots \\ 2.2727\ldots & -1.578\ldots \end{bmatrix} 10^4/s$$

(I used wolframalpha.com...)

$$B \approx \begin{bmatrix} -0.940088 & -0.244221 \\ 0.340932 & -0.969720 \end{bmatrix} \qquad D \approx \begin{bmatrix} -78451.8/s & 0 \\ 0 & -10059/s \end{bmatrix}$$

$$B^{-1} \approx \begin{bmatrix} -0.974706 & -0.342685 \\ 0.245477 & -0.944922 \end{bmatrix}$$

Let $\frac{1}{\tau_\alpha} = \frac{784518}{\text{second}}$ & $\frac{1}{\tau_\beta} = \frac{10059}{\text{second}}$

At the moment when we measured

$16mA = I_1 + I_2 = -\frac{dQ_2}{dt}$, we now have the equation $-16mA = dQ_2/dt$ **which** we can solve for t.

Once we have t, we can find Q_1 & Q_2, and then (finally!)

$U_1 = \frac{1}{2}\frac{Q_1^2}{5.0\mu F}$ & $U_2 = \frac{1}{2}\frac{Q_2^2}{7.2\mu F}$

at time t.

So, we need to solve ①②③, which is actually a math problem a little too advanced for this course, so I will show you the math and you won't feel bad if it's totally new to you.

$$\frac{d}{dt}\begin{bmatrix} Q_1 \\ Q_2 \end{bmatrix} = \underbrace{\begin{bmatrix} -\tau_{11}^{-1} & \tau_{21}^{-1} \\ \tau_{12}^{-1} & -\tau_{22}^{-1} \end{bmatrix}}_{A}\begin{bmatrix} Q_1 \\ Q_2 \end{bmatrix} = A\begin{bmatrix} Q_1 \\ Q_2 \end{bmatrix}$$

$$\Rightarrow \begin{bmatrix} Q_1 \\ Q_2 \end{bmatrix} = e^{At}\begin{bmatrix} Q_{10} \\ Q_{20} \end{bmatrix} \quad \text{where} \begin{cases} Q_{10} = 0 \\ Q_{20} = 1.08 \cdot 10^{-5} C \end{cases}$$

$$\begin{bmatrix}Q_1\\Q_2\end{bmatrix} = B^{-1}\begin{bmatrix} e^{-t/\tau_\alpha} & 0 \\ 0 & e^{-t/\tau_\beta}\end{bmatrix} B \begin{bmatrix}Q_{10}\\Q_{20}\end{bmatrix}$$

$$\Rightarrow \frac{d}{dt}\begin{bmatrix}Q_1\\Q_2\end{bmatrix} = B^{-1}\begin{bmatrix} \left(-\frac{1}{\tau_\alpha}\right)e^{-t/\tau_\alpha} & 0 \\ 0 & \left(-\frac{1}{\tau_\beta}\right)e^{-t/\tau_\beta}\end{bmatrix} B \begin{bmatrix}Q_{10}\\Q_{20}\end{bmatrix}$$

~~$\Rightarrow \underbrace{-16\,mA}_{-1.6\cdot 10^{-2}A} = \frac{dQ_2}{dt} \approx -0.27295\left(e^{-7845 t/second}\right)A - 0.025728\left(e^{-10059 t/second}\right)A$~~

~~We can't solve this equation with a formula but a computer/calculator can estimate the solution: $t = 2.761 \cdot 10^{-4}\,s$~~

$\Rightarrow -16\,mA = -1.6\cdot 10^{-2}A = dQ_2/dt$

& $dQ_2/dt \approx (9.8857\cdot 10^{-2}\,A)e^{-7845 t/second}$

$\quad - (1.02158\cdot 10^{-1}A)\left(e^{-10059 t/second}\right)$

Solving $-16\,mA = dQ_2/dt$ for t requires a calculator or computer to estimate t; there is no formula for t.

$\boxed{t \approx 4.13213 \cdot 10^{-4}\,s}$

(There is also a negative numerical solution which we reject because t must be positive because we're consider a time after we removed the battery.)

Plugging in t into Q_1 & Q_2, we find

$\boxed{Q_1 \approx 4.00592\cdot 10^{-8}C \;\&\; Q_2 \approx 1.59062\cdot 10^{-7}C}$

$\Rightarrow U_1 = \frac{Q_1^2}{2(5.0\mu F)} = \boxed{1.6\cdot 10^{-10}J}$ & $U_2 = \frac{Q_2^2}{2(7.2\mu F)} = \boxed{1.8\cdot 10^{-9}J}$