Oct/05/2010

Limits 3.3

$$\lim_{x \to c} f(x) = L$$

- "The limit of $f(x)$ as x approches c is L".
- As $x \to c$, $f(x) \to L$
- $f(x) \to L$ as $x \to c$

Definition
- $c \approx x \neq c \Rightarrow f(x) \approx L$
- Whenever x is infinitely close but not equal to, $f(x)$ is infinitely close to L

Equivalent definition:
- $x \neq c = st(x) \Rightarrow f(x) \approx L$
- $x \neq c = st(x) \Rightarrow st(f(x)) = L$
- $0 \neq \varepsilon \approx 0 \Rightarrow f(c+\varepsilon) \approx L$
- $\boxed{0 \neq \varepsilon \approx 0 \Rightarrow st(f(c+\varepsilon)) = L}$ ← Best for computing L

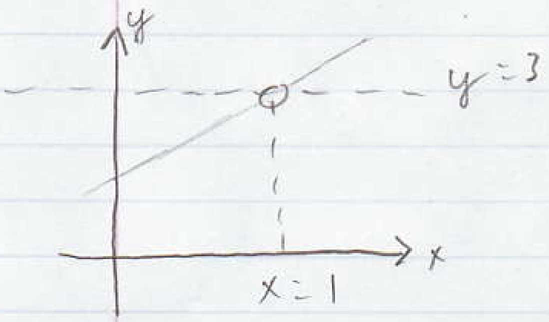
Example
$$\lim_{x \to 1} \frac{x^2+x-2}{x-1} = ?$$

Let $0 \neq \varepsilon \approx 0$
$$f(1+\varepsilon) = \frac{(1+\varepsilon)^2+(1+\varepsilon)-2}{(1+\varepsilon)-1}$$
$$= \frac{(1+2\varepsilon+\varepsilon^2)+\varepsilon-1}{\varepsilon}$$

$$\frac{\varepsilon^2+3\varepsilon}{\varepsilon} = \frac{\varepsilon(\varepsilon+3)}{\varepsilon} = \varepsilon+3$$
$$st(f(1+\varepsilon)) = st(\varepsilon+3) = 0+3 = 3$$

$$\lim_{x \to 1} \frac{x^2+x-2}{x-1} = 3$$

Note $f(1)$ $\frac{1^2+1-2}{1-1} = \frac{0}{0}$ is undefined



$$\lim_{x \to 1} \underbrace{\frac{x^2+x-2}{x-1}}_{f(x)} = ?$$

$$f(x) = \frac{(x-1)(x+2)}{x-1}$$

Let $0 \neq \varepsilon \simeq 0$

$$f\,\frac{(1+\varepsilon-1)\,(1+\varepsilon+2)}{1+\varepsilon-1}$$

$$= \frac{\varepsilon(3+\varepsilon)}{\varepsilon} = 3+\varepsilon$$

$$\Rightarrow st(f(1+\varepsilon)) = 3+0 = 3 \Rightarrow \lim_{x \to 1} f(x) = 3$$

Derivatives are limits

$$f'(c) = st\left(\frac{f(c+\Delta x) - f(c)}{\Delta x}\right)$$

For all $0 \neq \Delta x \simeq 0$

Same as: $f(c) = \lim_{\Delta x \to 0} \frac{f(c+\Delta x) - f(c)}{\Delta x}$

$$\lim_{y \to 0} \frac{(5+y)^3 - 125}{y} = ?$$

where $f(x) = x^3$

$f'(x) = 3x^2$

$$= \lim_{\Delta x \to 0} \frac{(5+\Delta x)^3 - 125}{\Delta x}$$

$$= \lim_{\Delta x \to 0} \frac{(5+\Delta x)^3 - 5^3}{\Delta x} = f'(5) = 3 \cdot 5^2 = 3 \cdot 25 = \boxed{75}$$

There are three ways a limit can

$\lim_{x \to c} f(x)$ could fail to exist:

① There is an ε such that $0 \neq \varepsilon \approx 0$ and $f(c+\varepsilon)$ is undefined.

② There is an ε such that $0 \neq \varepsilon \approx 0$ and $f(c+\varepsilon)$ is infinite

③ There is an ε_1 & an ε_2 such that $0 \neq \varepsilon_1 \approx 0$ & $0 \neq \varepsilon_2 \approx 0$ & $st(f(c+\varepsilon_1)) \neq st(f(c+\varepsilon_2))$

Example

① $f(x) = \sqrt{x}$ Let $0 > \varepsilon \approx 0$. $\sqrt{0+\varepsilon} = \sqrt{\varepsilon}$ is undefine

So, $\lim_{x \to 0} \sqrt{x}$ does not exist

② $f(x) = \frac{1}{x-3}$ Let $0 \neq \varepsilon \approx 0$. $f(3+\varepsilon) = \frac{1}{3+\varepsilon-3} = \frac{1}{\varepsilon}$ Infinite

So $\lim_{x \to 3} \frac{1}{x-3}$ does not exist

$st(f(0+\varepsilon_1)) = st(1) = 1$
$\neq -1 = st(-1) = st(f(0+\varepsilon_2))$

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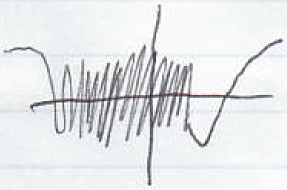
$$f(x) \quad \frac{|x|}{x}. \begin{cases} \text{Let } 0 < \varepsilon \simeq 0 \; f(0+\varepsilon) = \dfrac{|0+\varepsilon|}{0+\varepsilon_1} = \dfrac{\varepsilon_1}{\varepsilon_1} = 1 \\ \text{Let } 0 > \varepsilon_2 \simeq 0. \; f(0+\varepsilon_2) \; \dfrac{-\varepsilon_2}{\varepsilon_2} = -1 \end{cases}$$

$|x|$ = distance from x to 0
$|-5| = 5 = -(-5)$ $\qquad |5| = 5$

Picture of
$\lim\limits_{x \to 0} \cos\left(\frac{1}{x}\right)$ ONE



$\lim\limits_{x \to 0} x \cos\left(\frac{1}{x}\right) = 0$