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Definition: c is a <u>critical point</u> of $f(x)$ if $f'(c)=0$ or if $f'(c)$ does not exist

critical points and end points are the usual places for maximums and minimums to occur.

How to find the max. and min. values of $f(x)$ <u>over $[a,b]$</u> <u>when f is continuous on $[a,b]$</u>.

step 0: check that f is really continuous on $[a,b]$

* the EVT needs cont. to guarantee that the max./min. exist

step 1: find all critical points c in (a,b)

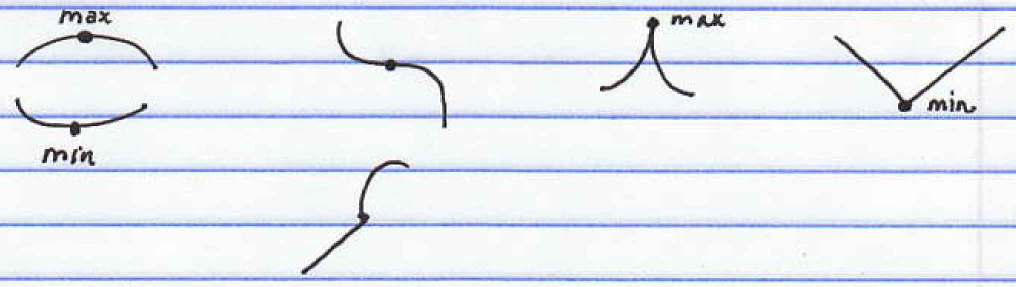
step 2: make a list of values: $f(a)$, $f(b)$, and $f(c)$ for all c's you've found

step 3: the greatest value is the max. value of $f(x)$ over $[a,b]$ – the least value on the list is the min. value of $f(x)$ over $[a,b]$.

Why does this work? See 3.5 & pictures:

critical pts.

$f'(c)=0$ $f'(c)$ DNE



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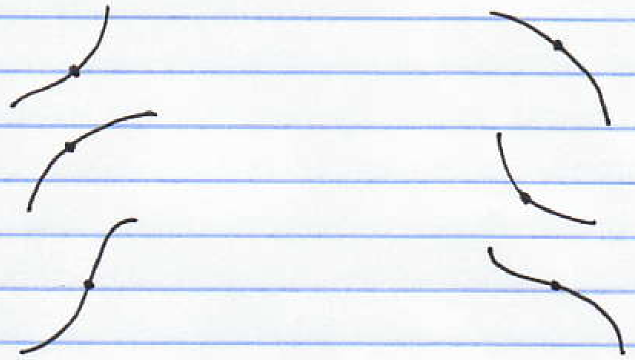
c not critical, not endpoint

$f'(c) > 0$ $\qquad\qquad$ $f'(c) < 0$



* never min or max of a noncritical point in the interior

Example: find the max. & min. values of $f(x) = x^3 - x$ over $[-1, 2]$

step 0: f is cts. everywhere

step 1: $f'(x) = 3x^2 - 1$ exists everywhere

* could it be 0? $\quad 0 = 3x^2 - 1 \iff 1 = 3x^2 \iff \frac{1}{3} = x^2 \iff \pm\frac{1}{\sqrt{3}} = x$

$\pm\frac{1}{\sqrt{3}} = \pm 0.58\ldots$ both in $(-1, 2)$

step 2:

$f(-1) = (-1)^3 - (-1) = -1 + 1 = 0$

$f(2) = 2^3 - 2 = 8 - 2 = 6 \quad \leftarrow$ max

$f\left(\frac{1}{\sqrt{3}}\right) = \left(\frac{1}{\sqrt{3}}\right)^3 - \frac{1}{\sqrt{3}} = \frac{1}{3\sqrt{3}} - \frac{1}{\sqrt{3}} = -0.3849\ldots \quad \leftarrow$ min

$f\left(-\frac{1}{\sqrt{3}}\right) = \left(\frac{-1}{\sqrt{3}}\right)^3 - \left(\frac{-1}{\sqrt{3}}\right) = \frac{-1}{3\sqrt{3}} + \frac{1}{\sqrt{3}} = 0.3849\ldots$

The max value of $x^3 - x$ over $[-1, 2]$ is 6 and min. value of $x^3 - x$ over $[-1, 2]$ is $\frac{1}{3\sqrt{3}} - \frac{1}{\sqrt{3}}$

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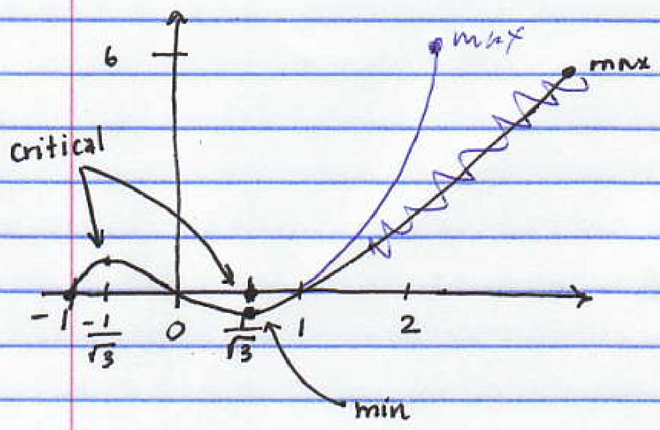

Example: Find the min. & max. values of
$f(x) = \sqrt[3]{x^2}$ over $[-8, 8]$.

$\sqrt[3]{x^2}$ is cts. everywhere. ✓

$$f(x) = x^{2/3} \Rightarrow f'(x) = 2/3\, x^{2/3 - 1}$$
$$= \tfrac{2}{3} x^{-1/3}$$
$$= \frac{2}{3x^{1/3}}$$
$$= \frac{2}{3\sqrt[3]{x}}$$

$f'(0) = \dfrac{2}{3\sqrt[3]{0}}$: D.N.E.

0 is in $[-8,8]$

$$0 = \frac{2}{3\sqrt[3]{x}} \Rightarrow 3\sqrt[3]{x} \cdot 0 = 2 \Rightarrow 0 \neq 2 \text{ (impossible)}$$

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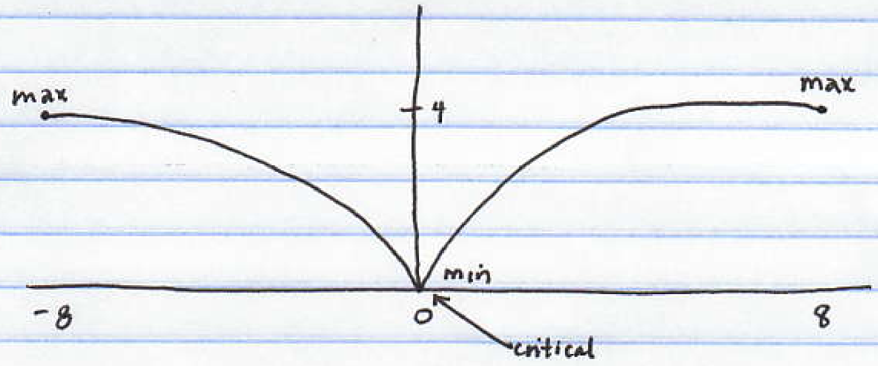
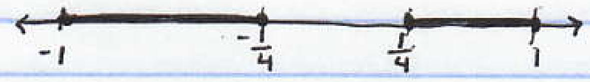
$f(-8) = \sqrt[3]{(-8)^2} = \sqrt[3]{64} = 4$
$f(8) = \sqrt[3]{(8)^2} = \sqrt[3]{64} = 4$ } max.
$f(0) = \sqrt[3]{0^2} = 0$ ← min.

The min. value of $\sqrt[3]{x^2}$ over $[-8,8]$ is 0
" max " " " " " " 4.



Example: Find the max. & min. values of $f(x) = x^2 + \frac{1}{x}$
over $[-1, -\frac{1}{4}] \cup [\frac{1}{4}, 1]$



The same method works; just check continuity for each interval & check <u>all</u> endpoints

f is cts. everywhere but 0 — but 0 is not in either interval

f is cts. on $[-1, \frac{1}{4}]$
f is cts on $[\frac{1}{4}, 1]$

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$f'(x) = 2x - \frac{1}{x^2}$ — $f'(x)$ exists everywhere but 0.
— 0 is a critical point but 0 is not in $(-1, \frac{1}{4})$ or $(\frac{1}{4}, 1)$.

$$f'(x) = \frac{2x \cdot x^2}{x^2} - \frac{1}{x^2} = \frac{2x^3-1}{x^2}$$

$$0 = \frac{2x^3-1}{x^2}$$

$$0 = 0 \cdot x^2 = 2x^3 - 1$$

$$1 = 2x^3$$

$$\frac{1}{2} = x^3$$

$$\frac{1}{\sqrt[3]{2}} = x$$

$t = 0.7937\ldots$ is in $(\frac{1}{4}, 1)$

$f(-1) = (-1)^2 + \frac{1}{-1} = 1 - 1 = 0$
$f(-\frac{1}{4}) = (-\frac{1}{4})^2 + \frac{1}{-1/4} = \frac{1}{16} - 4 = -\frac{63}{16}$ or -3.9375 ← min
$f(\frac{1}{4}) = (\frac{1}{4})^2 + \frac{1}{1/4} = \frac{1}{16} + 4 = \frac{65}{16}$ or 4.0625 ← max
$f(1) = (1)^2 + \frac{1}{1} = 1 + 1 = 2$
$f(\frac{1}{\sqrt[3]{2}}) = (\frac{1}{\sqrt[3]{2}})^2 + \frac{1}{1/\sqrt[3]{2}} = \frac{1}{\sqrt[3]{4}} + \sqrt[3]{2} = 1.889\ldots$