Oct 19, 10.

(1)

Find the maximum and minimum values of $\frac{1}{x^2+1}$ over $(-\infty,\infty)$, if they exist.

- $x^2+1 \geq 0+1 \geq 1$, so it is cts. everywhere.
- Let H be positive infinite. ((closed) interval)
- Approximate $(-\infty,\infty)$ with $[-H,H]$

[←...... real line ......→] from $-H$ to H

1) Find critical points

$$f'(x) = \frac{-(x^2+1)'}{(x^2+1)^2} = \frac{-2x}{(x^2+1)^2} \qquad \left(\frac{1}{g}\right)' = \frac{-g'}{g^2}$$

- $x^2+1 \geq 1$, so $(x^2+1)^2 \geq 1^2 > 0$. No division by zero, so it is defined everywhere.

$$0 = \frac{-2x}{(x^2+1)^2} \Rightarrow (x^2+1)^2 \cdot 0 = -2x$$

$$\Rightarrow 0 = -2x = \frac{0}{-2} = x \Rightarrow 0 \leftarrow \text{Only critical point.}$$

2) Table of values

	x	$f(x)$
• Sub given	$-H$	$1/((-H)^2+1) = 1/(H^2+1)$ ← Least value
points and	0	$1/(0^2+1) = 1/1 = 1$ ← Greatest value
critical points	H	$1/(H^2+1)$ ← Least value

- $\frac{1}{H^2+1} = \frac{\text{finite}}{\text{infinite}} =$ ~~[illegible]~~ Infinitesimal.
- Least value appears twice in this problem.

*The greatest value, 1, is at $x=0$, and 0 is real, so 1 is the maximum value over the whole hyperreal line $(-\infty,\infty)$. The least value on the list does NOT occur at a real, so $f(x)$ does NOT have a minimum value over $(-\infty,\infty)$. The least values don't count because they are just approximations.

* If a real function has a min/max, it happens at a REAL x-value.

(2) Find the min/max values of $f(x) = \frac{1}{x^2}$ over $[-4,0) \cup (0,3)$, if they exist.

0) f is cts. everywhere but 0: $\frac{1}{0^2}$ so f is cts on $[-4,0)$ and $(0,3)$.

1) Approximate $[-4,0) \cup (0,3)$ by $[-4,-\varepsilon] \cup [\varepsilon, 3-\varepsilon]$ where $0 < \varepsilon \approx 0$.

2) Find ~~crit~~ critical points:

• Taking $f(x) = x^{-2} \Rightarrow f'(x) = -2x^{-3}$

derivative $f'(x) = \frac{-2}{x^3}$

$f'(0)$ is NOT defined: $\frac{-2}{0^3}$

• However, 0 is NOT in $[-4,-\varepsilon] \cup [\varepsilon, 3-\varepsilon]$, so ignore.

$\frac{-2}{x^3} = 0 \Rightarrow -2 = 0 \cdot x^3 = 0$ (impossible)

So, there are no critical points in our intervals

3)

x	$1/x^2$	
-4	$1/16$	← Least
$-\varepsilon$	$1/\varepsilon^2$	} Infinite, positive. ← Both greatest.
ε	$1/\varepsilon^2$	
$3-\varepsilon$	$1/(3-\varepsilon)^2 \approx 1/(3-0)^2 = 1/9$	

* $1/x^2$ has minimum value $1/16$ over $[-4,0) \cup (0,3)$, but NO maximum value over $[-4,0) \cup (0,3)$

(3) Find the max/min values of $f(x) = x^2(3x^2-4x-12)$ over $(-2,3]$.

0) f is cts. everywhere

1) $(-2,3] \longrightarrow [-2+\varepsilon, 3]$

2) $f(x) = 3x^4-4x^3-12x^2$

$f'(x) = 12x^3-12x^2-24x$ *There is a common factor!

$= 12x(x^2-x-2)$ *Quadratic!

$= 12(x+1)(x-2)$ *Defined everywhere!!!

$0 = 12x(x+1)(x-2)$

$\underline{0=12x}$ OR $\underline{0=x+1}$ OR $\underline{0=x-2}$

$[0=x \; , \; -1=x \; , \; 2=x]$ = Critical points

Skip algebra

x	$f(x)$	
$-2+\varepsilon$	$f(-2+\varepsilon) \approx f(-2) = 32$	by continuity at -2; Greatest
-1	-5	
0	0	
2	-32	Least
3	27	

* Over $(-2,3]$, $f(x)$ has min. value -32, but no max. value.