$\int_P F\,dG$. What does that mean?

P is a path. F is a function that works like this:

$$\text{position } x \text{ on path} \longrightarrow \boxed{F} \longrightarrow \text{real number } F(x)$$

G is another function like F:

$$\text{position } x \text{ on path} \longrightarrow \boxed{G} \longrightarrow \text{real number } G(x)$$

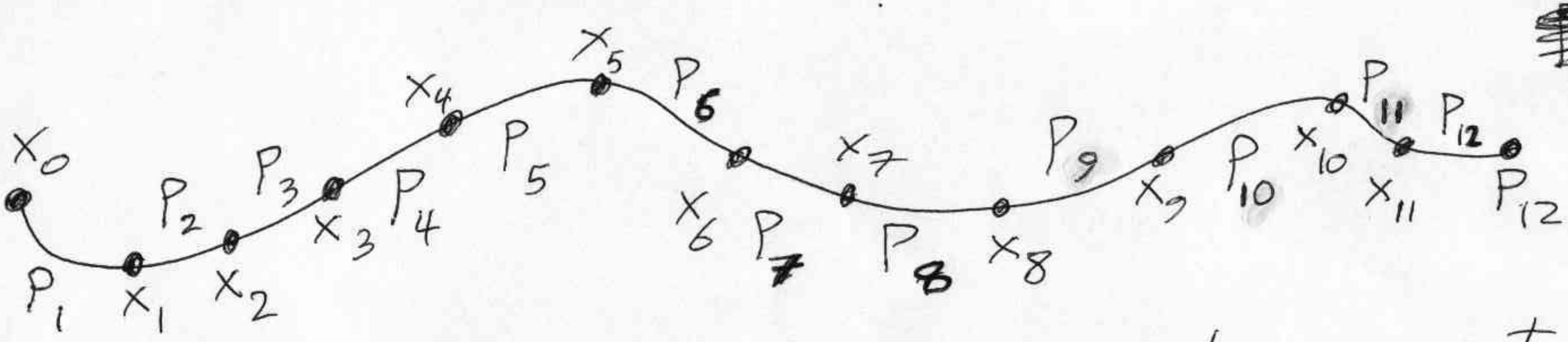
"$\int$" means "limit of finite sums"

"d" means "difference of" or "change of"

Don't ask, "what is $\int_P F \, dG$?"

Ask how to approximate it. 4 steps:

(1) Break up path P into large number of small segments $P_1, \ldots, P_n$ with endpoint positions $x_0, \ldots, x_n$. ($n=12$ example below.)



(2) Estimate "average" value of F on each segment. (Maybe use an endpoint value; maybe use average of both endpoint values (that's the trapezoid rule); maybe use the midpoint value of F; etc.)

(3) Estimate change of G from start to end of each segment. (Simplest and best thing: $G(x_i) - G(x_{i-1})$, the actual change, unless G is a really complicated function, and you need a simpler, approximate formula for the change in G from x_{i-1} to x_i.)

(4) Multiply and add:

(estimate of F on P_1) $\cdot$ (estimated $G(x_1) - G(x_0)$)

+ (estimate of F on P_2) $\cdot$ (estimated $G(x_2) - G(x_1)$)

+ (estimate of F on P_3) $\cdot$ (estimated $G(x_3) - G(x_2)$)

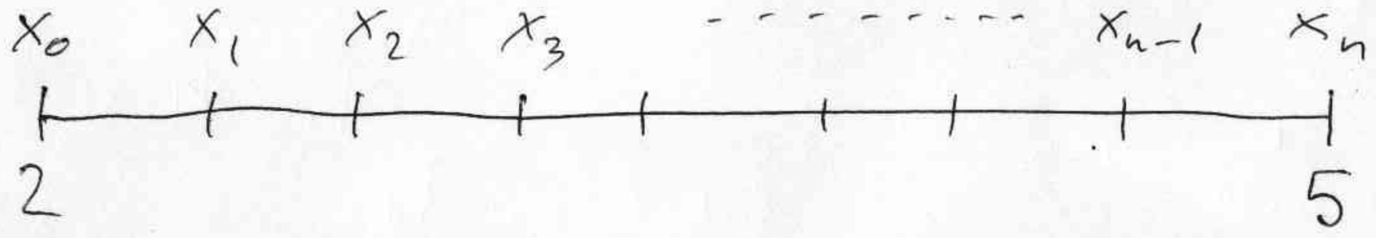
$\vdots$

+ (estimate of F on P_n) $\cdot$ (estimated $G(x_n) - G(x_{n-1})$)

Example: $F(x)=x^2$, $G(x)=x$, $P=[2,5]$

= path from 2 to 5. "$\int_2^5$" means "$\int_P$" then.

$$\int_P F\,dG \approx \underbrace{\left(\frac{x_0+x_1}{2}\right)^2}_{\text{midpoint rule}}\underbrace{(x_1-x_0)}_{\text{exact } dG} + \cdots + \underbrace{\left(\frac{x_n+x_{n-1}}{2}\right)^2}_{\text{midpoint rule}}\underbrace{(x_n-x_{n-1})}_{\substack{dG\\ \text{exact}}}$$



$\int_2^5 x^2\,dx$ is the above $\int_P F\,dG$.

The above midpoint rule is one of many ways to estimate $\int_2^5 x^2\,dx$.

But we know $\int_2^5 x^2\,dx = \left.\frac{x^3}{3}\right|_2^5 = \frac{125-8}{3} = \frac{117}{3}$

$= 39$ exactly! How do we know?

Because of a clever estimate! Consider

$\int_Q H\,dK$ where $Q = [2,5]$ = path from 2 to 5,

$H = 1$, and $K(x) = x^3/3$.

$$\int_Q H\,dK = \int_{x=2}^{x=5} 1\,d(x^3/3) = \int_{x=2}^{x=5} d(x^3/3)$$

$$\approx \boxed{\left(\frac{x_1^3}{3} - \frac{x_0^3}{3}\right) + \left(\frac{x_2^3}{3} - \frac{x_1^3}{3}\right) + \cdots + \left(\frac{x_n^3}{3} - \frac{x_{n-1}^3}{3}\right)}$$

All but two terms cancel out!

$$= \frac{x_n^3}{3} - \frac{x_0^3}{3} = \frac{5^3}{3} - \frac{2^3}{3} = 39.$$

What does $\int_{x=2}^{x=5} d(x^3/3)$ have to do with $\int_2^5 x^2\,dx$? If $x_i - x_{i-1}$ is small,

then $\dfrac{x_i^3/3 - x_{i-1}^3/3}{x_i - x_{i-1}} \approx x_{i-1}^2$.

(That's the definition of derivative! $(x^3/3)' = x^2$,

so $x_{i-1}^2 = \lim_{h\to\infty} \dfrac{(x_{i-1}+h)^3/3 - x_{i-1}^3/3}{h} = \lim_{x_i \to x_{i-1}} \dfrac{x_i^3/3 - x_{i-1}^3/3}{x_i - x_{i-1}}$

if $h = x_i - x_{i-1}$.)

Hence, $x_i^3/3 - x_{i-1}^3/3 \approx x_{i-1}^2 (x_i - x_{i-1})$.

More elegantly, we write "$d(x^3/3) = x^2\,dx$".

because we almost always prefer to use "the approximation $x^2 \cdot (\text{change in } x)$ over the exact change in $x^3/3$.

If we really want the exact change, we write "$\Delta(x^3/3)$." (Greek D in place of Roman d.)

In the limit $n \to \infty$, $dx \to 0$, the approximation error of $\Delta(x^3/3) \approx x^2\,dx$ goes away. Therefore,

$$39 = \int_{x=2}^{x=5} d(x^3/3) = \int_2^5 x^2\,dx.$$

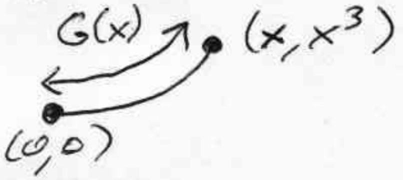
That's the 2nd Fundamental Theorem of Calculus.

Example: $F = 2\pi r$; $r(x) = 2 - x^3$;

$P = [0, 1]$ (path from 0 to 1); $G(x)$

$=$ arc length from $(0,0)$ to (x, x^3)

along the curve $y = x^3$.



(straight-line approximation)

$$dG \approx ds \overset{}{=} \sqrt{dx^2 + dy^2} = \sqrt{1 + (dy/dx)^2}\, dx = \sqrt{1 + (3x^2)^2}\, dx$$

(definition of ds) (using $dy = (x^3)' dx$ definition)

$$\int_P F\, dG = \int_0^1 (2 - x^3)\sqrt{1 + 9x^4}\, dx$$ because

in limit $n \to \infty$, $dx \to 0$, the $dG \approx ds$ approximation errors go away. (More precisely, $\frac{ds}{dG} \to 1$ as $dx \to 0$.)