A proof of $\lim\limits_{n \to \infty} \left(1 + \frac{c}{n}\right)^n = \sum\limits_{n=0}^{\infty} \frac{c^n}{n!}$:

In general, $(a+b)^n = a^n + n a^{n-1} b$

$$+ \frac{n(n-1)}{2} a^{n-2} b^2 + \frac{n(n-1)(n-2)}{6} a^{n-3} b^3$$

$$+ \frac{n(n-1)(n-2)(n-3)}{24} a^{n-4} b^4$$

$$+ \cdots + \frac{n(n-1)(n-2) \cdots (n-(k-1))}{k!} a^{n-k} b^k$$

$$+ \cdots + \frac{n(n-1)(n-2) \cdots (3)(2)}{(n-1)!} a b^{n-1}$$

$$+ \frac{n(n-1)(n-2) \cdots (3)(2)(1)}{n!} b^n$$

$$= \sum_{k=0}^{n} \binom{n}{k} a^{n-k} b^k \quad \text{where} \quad \binom{n}{k} = \frac{n(n-1) \cdots (n-(k-1))}{k!},$$

$$= \frac{\overbrace{n(n-1) \cdots (n-(k-1))(n-k)(n-(k+1)) \cdots (2)(1)}^{n!}}{k! \underbrace{(n-k)(n-(k+1)) \cdots (2)(1)}_{(n-k)!}},$$

So, $\left(1+\frac{c}{n}\right)^n = \sum_{k=0}^{n} \frac{n!}{k!(n-k)!} 1^{n-k} \left(\frac{c}{n}\right)^k$

$$= \sum_{k=0}^{n} \left(\frac{c^k}{k!} \cdot \frac{n!}{(n-k)!\, n^k} \right)$$

$$= \sum_{k=0}^{n} \left(\frac{c^k}{k!} \cdot \frac{\overbrace{1 \cdot 2 \cdot 3 \cdot \cdots \cdot (n-k) \cdot (n-(k-1)) \cdot \cdots \cdot (n)}^{n!}}{\underbrace{1 \cdot 2 \cdot 3 \cdot \cdots \cdot (n-k)}_{(n-k)!} \cdot \underbrace{(n) \cdot \cdots \cdot (n)}_{n^k}} \right)$$

$$= \sum_{k=0}^{n} \left(\frac{c^k}{k!} \cdot \frac{n-(k-1)}{n} \cdot \frac{n-(k-2)}{n} \cdot \cdots \cdot \frac{n-1}{n} \cdot \frac{n}{n} \right)$$

$$= \sum_{k=0}^{n} \left[\frac{c^k}{k!} \cdot \underbrace{\left(1 - \frac{k-1}{n}\right) \cdot \left(1 - \frac{k-2}{n}\right) \cdot \cdots \cdot \left(1 - \frac{1}{n}\right) \left(1 - \frac{0}{n}\right)}_{\text{abbreviate as } \prod_{m=0}^{k-1} \left(1 - \frac{m}{n}\right)} \right]$$

$$= \sum_{k=0}^{N-1} \frac{c^k}{k!} \prod_{m=0}^{k-1} \left(1 - \frac{m}{n}\right) + \sum_{k=N}^{n} \frac{c^k}{k!} \prod_{m=0}^{k-1} \left(1 - \frac{m}{n}\right)$$

for every $N \leq n$.

To prove that $\lim_{m\to\infty}\left(1+\frac{c}{m}\right)^m = \sum_{k=0}^{\infty}\frac{c^k}{k!}$,

we will prove that, for every $\varepsilon > 0$,

$$\left|\lim_{m\to\infty}\left(1+\frac{c}{m}\right)^m - \sum_{k=0}^{\infty}\frac{c^k}{k!}\right| < \varepsilon.$$

Given $\varepsilon > 0$, choose N large enough that $\sum_{k=N}^{\infty}\frac{|c|^k}{k!} < \frac{\varepsilon}{4}$, which is possible because $\sum_{k=0}^{\infty}\frac{|c|^k}{k!}$ converges by the ratio test. Then

$$\left|\sum_{k=N}^{n}\frac{c^k}{k!}\prod_{m=0}^{k-1}\left(1-\frac{m}{n}\right)\right| \leq \sum_{k=N}^{n}\frac{|c|^k}{k!}\underbrace{\prod_{m=0}^{k-1}\left(1-\frac{m}{n}\right)}_{\leq 1}$$

$$\leq \sum_{k=N}^{n}\frac{|c|^k}{k!} < \sum_{k=N}^{\infty}\frac{|c|^k}{k!} < \frac{\varepsilon}{4} \text{ for all } n \geq N.$$

Now choose n large enough that

$$\left|\sum_{k=0}^{N-1}\frac{c^k}{k!}-\sum_{k=0}^{N-1}\frac{c^k}{k!}\prod_{m=0}^{k-1}\left(1-\frac{m}{n}\right)\right|<\frac{\varepsilon}{4}$$

and $$\left|\left(1+\frac{c}{n}\right)^n-\lim_{m\to\infty}\left(1+\frac{c}{m}\right)^m\right|<\frac{\varepsilon}{4},$$

which is possible because

$$\lim_{n\to\infty}\sum_{k=0}^{N-1}\frac{c^k}{k!}\prod_{m=0}^{k-1}\left(1-\frac{m}{n}\right)=\sum_{k=0}^{N-1}\frac{c^k}{k!}\underbrace{\prod_{m=0}^{k-1}(1-0)}_{(1-0)^k=1}$$

$$=\sum_{k=0}^{N-1}\frac{c^k}{k!}.$$ Finally, ~~the distance~~

$$\left|\sum_{k=0}^{\infty}\frac{c^k}{k!}-\sum_{k=0}^{N-1}\frac{c^k}{k!}\right|=\left|\sum_{k=N}^{\infty}\frac{c^k}{k!}\right|$$

$$\leq\sum_{k=N}^{\infty}\frac{|c|^k}{k!}<\frac{\varepsilon}{4}$$ by our earlier choice

of N. Combining four estimates,

we obtain $\left|\lim_{m\to\infty}\left(1+\frac{c}{m}\right)^m - \sum_{k=0}^{\infty}\frac{c^k}{k!}\right| < \varepsilon$

because the distance between $\lim_{m\to\infty}\left(1+\frac{c}{m}\right)^m$ and $\sum_{k=0}^{\infty}\frac{c^k}{k!}$ is at most the distance between $\lim_{m\to\infty}\left(1+\frac{c}{m}\right)^m$ and $\left(1+\frac{c}{n}\right)^n$, plus the distance between $\left(1+\frac{c}{n}\right)^n$ and $\sum_{k=0}^{N-1}\frac{c^k}{k!}\prod_{m=0}^{k-1}\left(1-\frac{m}{n}\right)$ (this distance is $\left|\sum_{k=N}^{n}\frac{c^k}{k!}\prod_{m=0}^{k-1}\left(1-\frac{m}{n}\right)\right|$), plus the distance between $\sum_{k=0}^{N-1}\frac{c^k}{k!}\prod_{m=0}^{k-1}\left(1-\frac{m}{n}\right)$ and $\sum_{k=0}^{N-1}\frac{c^k}{k!}$, plus the distance between $\sum_{k=0}^{N-1}\frac{c^k}{k!}$ and $\sum_{k=0}^{\infty}\frac{c^k}{k!}$. These last four distances are each $< \frac{\varepsilon}{4}$, so the total distance is $< \varepsilon$.

Since our argument works for any positive ε, we must have

$$\lim_{m \to \infty} \left(1 + \frac{c}{m}\right)^m = \sum_{k=0}^{\infty} \frac{c^k}{k!}.$$

Therefore,

$$e^c = \sum_{k=0}^{\infty} \frac{c^k}{k!}.$$