Summary: $\sum_{n=0}^{\infty} \binom{p}{n} x^n$ is the binomial series. Whenever it converges, it converges to $(1+x)^p$ (though I will not prove that here).

In this note I will determine the intervals of convergence for $\sum_{n=0}^{\infty} \binom{p}{n} x^n$, including proofs (except for $0<p<1$). Let $I(p)$ be the interval of convergence for a given $p \in \mathbb{R}$.

$p = 0, 1, 2, 3, 4, \ldots \Rightarrow I(p) = (-\infty, \infty)$

$0 < p \neq 1, 2, 3, 4, \ldots \Rightarrow I(p) = [-1, 1]$

$-1 < p < 0 \Rightarrow I(p) = (-1, 1]$

$p \leq -1 \Rightarrow I(p) = (-1, 1)$

Binomial series: For $|x|<1$, we have, for all real p, $$(1+x)^p = \sum_{n=0}^{\infty} \binom{p}{n} x^n$$ where $\binom{p}{0}=1$,

$$\binom{p}{1} = p, \quad \binom{p}{2} = \frac{p(p-1)}{2}, \quad \binom{p}{3} = \frac{p(p-1)(p-2)}{6}, \ \ldots$$

$$\binom{p}{n} = \frac{p(p-1)(p-2)\cdots(p-(n-1))}{n!} = \prod_{m=1}^{n} \frac{p-(m-1)}{m}, \ \ldots$$

The series $\sum_{n=0}^{\infty} \binom{p}{n} x^n$ converges by the ratio test for if $|x|<1$, and diverges by the ratio test for $|x|>1$. But what if $|x|=1$? That is, does $\sum_{n=0}^{\infty} \binom{p}{n}$ converge? Does $\sum_{n=0}^{\infty} \binom{p}{n}(-1)^n$ converge?

(Minor technicality: actually, if $p \in \{0, 1, 2, 3, \ldots\}$, then the binomial series converges for all x because

$$0 = \binom{p}{p+1} = \binom{p}{p+2} = \binom{p}{p+3} = \cdots.)$$

$$\binom{p}{n} = \prod_{m=1}^{n} \frac{p-(m-1)}{m} = \prod_{m=1}^{n}\left(\frac{p}{m} - 1 + \frac{1}{m}\right) = \prod_{m=1}^{n}\left((-1)\left(1-\frac{p+1}{m}\right)\right)$$

$$= (-1)^n \prod_{m=1}^{n}\left(1-\frac{p+1}{m}\right).$$ For $m > p+1$, $1 > 1-\frac{p+1}{m} > 0$,

so, for $n > p+1$, $$\binom{p}{n} = (-1)^n \left(\prod_{m=1}^{\lfloor p+1\rfloor}\left(1-\frac{p+1}{m}\right)\right)\prod_{m=\lfloor p+1\rfloor+1}^{n}\left(1-\frac{p+1}{m}\right)$$

& the sequence $c_n = \prod_{m=\lfloor p+1\rfloor+1}^{n}\left(1-\frac{p+1}{m}\right)$ is $\searrow$ & positive, so

$$\sum_{n=0}^{\infty}\binom{p}{n} = \sum_{n=0}^{\lfloor p+1\rfloor}\binom{p}{n} + \prod_{m=1}^{\lfloor p+1\rfloor}\left(\left[1-((p+1)/m)\right]\right)\sum_{n=\lfloor p+1\rfloor+1}^{\infty}(-1)^n c_n$$

converges <u>if</u> $c_n \to 0$ by the A.S.T.

We will prove that if $p > -1$, then $c_n \to 0$ by proving that $\ln(c_n) \to -\infty$ (since then $c_n = e^{\ln(c_n)} \to 0$).

$\ln'(x) = \frac{1}{x}$, so $\ln'(1) = 1$, so $\lim\limits_{h \to 0} \frac{\ln(1+h) - \ln(1)}{h}$

$= 1$, so, for h small enough, $\frac{1}{2} \leq \frac{\ln(1+h) - \ln(1)}{h} \leq \frac{3}{2}$,

so for h small enough & negative $\frac{h}{2} \geq \ln(1+h) - \ln(1)$

$= \ln(1+h) - 0 = \ln(1+h)$. Therefore, for m large enough, say, $m \geq M$, $-\frac{p+1}{m}$ is small enough that $\frac{1}{2} \leq \frac{\ln(1-(p+1)/m)}{(p+1)/m} \leq \frac{3}{2}$. If $p > -1$, then $-(p+1)/m$ is negative, in which case

$-\frac{p+1}{2m} \geq \ln\left(1 - \frac{p+1}{m}\right)$. Therefore, $\ln(c_n) = \sum\limits_{m=\lfloor p+1 \rfloor + 1}^{n} \ln\left(1 - \frac{p+1}{m}\right)$

$= \sum\limits_{m=\lfloor p+1 \rfloor+1}^{M-1} \ln\left(1 - \frac{p+1}{m}\right) + \sum\limits_{m=M}^{n} \ln\left(1 - \frac{p+1}{m}\right) \leq \sum\limits_{m=\lfloor p+1 \rfloor+1}^{M-1} \ln\left(1 + \frac{p+1}{m}\right)$

$- \sum_{m=M}^{n} \frac{p+1}{2m}$. Since $\sum\limits_{m=M}^{\infty} \frac{1}{m} = \infty$, we have

$\ln(c_n) \to -\infty$, Therefore, $c_n \to 0$.

Therefore, $\boxed{\sum_{n=0}^{\infty} \binom{p}{n} \text{ converges for } p > -1.}$

Since $p < m-1 \Downarrow 0 > p-(m-1) \Downarrow |p-(m-1)| = -p+(m-1)$

If $p \le -1$, then $\left|\binom{p}{n}\right| = \prod_{m=1}^{n} \frac{-p+(m-1)}{m}$

$= \frac{(-p)(1-p)(2-p)\cdots(n-1-p)}{1 \cdot 2 \cdot 3 \cdot 4 \cdot \cdots \cdot n} \ge \frac{(1)(2)(3)\cdots(n)}{1 \cdot 2 \cdot 3 \cdot \cdots \cdot n}$

$= 1$ because $-p \ge 1$. Thus, $\binom{p}{n} \not\to 0$ if $p \le -1$. $\boxed{\text{Hence, } \sum_{n=0}^{\infty} \binom{p}{n} \text{ \& } \sum_{n=0}^{\infty} (-1)^n \binom{p}{n} \text{ diverge for } p \le -1.}$ The remaining case is $\sum_{n=0}^{\infty} \binom{p}{n}(-1)^n$

for $p > -1$, with $p \ne 0, 1, 2, 3, 4, 5, \ldots$

For $-1 < p < 0 < n$, $\binom{p}{n}(-1)^n = (-1)^n \prod_{m=1}^{n} \frac{p-(m-1)}{m}$

$$= \prod_{m=1}^{n}\left((-1)\left(\frac{p-(m-1)}{m}\right)\right) = \prod_{m=1}^{n} \frac{m-1-p}{m} = \frac{(-p)(1-p)(2-p)\cdots(n-1-p)}{1\cdot 2\cdot 3\cdot 4\cdots n}$$

$$= \frac{(-p)(1-p)}{1\cdot n}\cdot\frac{2-p}{2}\cdot\frac{3-p}{3}\cdot\cdots\cdot\frac{n-1-p}{n-1}$$

$$> \frac{(-p)(1-p)}{n}\cdot\frac{2}{2}\cdot\frac{3}{3}\cdot\cdots\cdot\frac{n-1}{n-1} = \frac{(-p)(1-p)}{n}$$ since $-p > 0$.

Therefore, $\sum_{n=0}^{\infty}\binom{p}{n}(-1)^n \geq \overbrace{\binom{p}{0}}^{1} + \sum_{n=1}^{\infty}\frac{\overbrace{(-p)}^{>0}\overbrace{(1-p)}^{>1}}{n} = \infty.$

$\boxed{\textbf{Thus, } \sum_{n=0}^{\infty}\binom{p}{n}(-1)^n \text{ diverges for } -1 < p < 0.}$

In the remaining case, $\sum_{n=0}^{\infty}\binom{p}{n}(-1)^n$ where $0 < p \neq 0, 1, 2, 3, \ldots$, if $n > p$, then, letting $q = \lfloor p+1 \rfloor < p+2$, we have

$$\binom{p}{n}(-1)^n = \frac{(0-p)(1-p)(2-p)(3-p)\cdots(n-1-p)}{1\cdot 2\cdot 3\cdot 4\cdots\cdots n}$$

$$= \frac{(0-p)(1-p)\cdots(q-3-p)(q-2-p)(q-1-p)(q-p)(q+1+p)\cdots(n-1-p)}{(n+2-q)(n+3-q)\cdots(n-1)\cdot(n)\cdot(1)\cdot(2)\cdot(3)\cdots(n+1-q)}$$

Since $q < p+2$, we have

$q-1-p < 1$ & $q-p < 2$ & $q+1+p < 3$ & ... & $n-1-p < n+1-q$,

so $\frac{q-1-p}{1}$, $\frac{q-p}{2}$, $\frac{q+1+p}{3}$, ..., $\frac{n-1-p}{n+1-q}$ are all < 1.

and are all > 0 because $q-1-p = \lfloor p+1 \rfloor - (1+p) > 0$.

Letting $(0-p)(1-p)\cdots(q-2-p) = A$, we have

$$\left|\binom{p}{n}(-1)^n\right| < \frac{|A|}{\underbrace{(n+2-q)\cdots(n)}_{q-1 \text{ terms}}}.$$

If $p > 1$, then $q \geq 3$.

Therefore, for $p > 1$, we have $\left|\binom{p}{n}(-1)^n\right| < \dfrac{|A|}{\underbrace{(n+2-q)\cdots(n)}_{q-1 \text{ terms}}}$

$$\le \frac{|A|}{(n+2-q)^{q-1}} \le \frac{|A|}{(n+2-q)^{3-1}} = \frac{|A|}{(n+2-q)^{2}}.$$

Since $\displaystyle\sum_{n=q-1}^{\infty} \frac{|A|}{(n+2-q)^2} = \sum_{n=1}^{\infty} \frac{|A|}{n^2} < \infty$, it follows

that $\displaystyle\sum_{n=0}^{\infty} \left|\binom{p}{n}(-1)^n\right| < \sum_{n=0}^{q-2} \left|\binom{p}{n}(-1)^n\right| + \sum_{n=q-1}^{\infty} \frac{|A|}{(n+2-q)^2} < \infty.$

Thus, $\displaystyle\sum_{n=0}^{\infty} \binom{p}{n}(-1)^n$ converges if $p > 1$.

The final, most delicate case is $\displaystyle\sum_{n=0}^{\infty} \binom{p}{n}(-1)^n$ for $0 < p < 1$. I do not know how to prove convergence without using lots of math beyond the Calculus level.

The current version (Oct 28, 2014) of the Wikipedia article "Binomial series" has a very clever proof that $\left|\binom{p}{n}\right| \le \frac{e^{p^2+p}}{n^{1+p}}$ for $p, n > 0$. However, this proof is not self-contained; it uses the arithmetic-mean-geometric-mean inequality

$$\frac{1}{n}\sum_{k=1}^{n} x_k \ge \left(\prod_{k=1}^{n} x_k\right)^{1/n} \quad \text{for } x_1, \dots, x_n \ge 0.$$

If you read that proof or accept on faith that $\left|\binom{p}{n}\right| \le \frac{e^{p^2+p}}{n^{1+p}}$ for $p, n > 0$, then it follows that

$\sum_{n=0}^{\infty} \binom{p}{n}$ & $\sum_{n=0}^{\infty} \binom{p}{n}(-1)^n$ converge (absolutely) for all $p > 0$,

because

$$\sum_{n=0}^{\infty} \left|\binom{p}{n}(-1)^n\right| = \sum_{n=0}^{\infty} \left|\binom{p}{n}\right| \le \sum_{n=0}^{1} \binom{p}{n} + e^{p^2+p} \sum_{n=2}^{\infty} \frac{1}{n^{p+1}}$$

$$\le 1 + p + e^{p^2+p} \int_1^{\infty} x^{-p-1}\,dx = 1 + p + e^{p^2+p} p^{-1} < \infty.$$