April 7, 2010

11.4 - Direct Comparison test

Suppose

$$0 \le a_0 \le b_0$$
$$0 \le a_1 \le b_1$$
$$0 \le a_2 \le b_2$$
$$\vdots$$
$$0 \le a_n \le b_n$$
$$\vdots$$

If $\sum_{n=0}^{\infty} a_n = \infty$ (diverges), then $\sum_{n=0}^{\infty} b_n \geq \sum_{n=0}^{\infty} a_n = \infty$ (diverges)

If $\sum_{n=0}^{\infty} b_n < \infty$ (converges), then $\sum_{n=0}^{\infty} a_n \le \sum_{n=0}^{\infty} b_n < \infty$ (converges)

$\sum_{n=1}^{\infty} \frac{1}{n^2+\sqrt{n}} \Rightarrow$ Converges: $0 \le \frac{1}{n^2+\sqrt{n}} \le \frac{1}{n^2}$ for all $n \geq 1 \rightarrow$

$$\Rightarrow 0 \le \sum_{n=1}^{\infty} \frac{1}{n^2+\sqrt{n}} \le \sum_{n=1}^{\infty} \frac{1}{n^2} = \frac{\pi^2}{6} < \infty$$

$\sum_{n=1}^{\infty} \frac{1}{n} = \infty \Rightarrow \lim_{n \to \infty} \frac{1}{n} = 0$ (diverges)

$\sum_{n=1}^{\infty} \frac{1}{n^2} = \frac{\pi}{6} < \infty \Rightarrow \sum_{n=1}^{\infty} \frac{1}{n^2} \le \frac{1}{1^2} + \int_1^{\infty} \frac{dx}{x^2} = \frac{1}{1} + 1 = 2$ (converges)

$$\sum_{n=3}^{\infty} \frac{\sqrt{n}}{(n-1)^{3/2}} \Rightarrow \text{compare to: } \sum_{n=1}^{\infty} \frac{1}{n^p} = \begin{cases} \infty : p \le 1 \\ \text{convergent} : p > 1 \end{cases}$$

For all

$$n \geq 3 : (n-1)^{3/2} \le n^{3/2} \Rightarrow \frac{\sqrt{n}}{(n-1)^{3/2}} \geq \frac{\sqrt{n}}{n^{3/2}} = \frac{n^{1/2}}{n^{3/2}} = \frac{1}{n} \Rightarrow$$

$$\sum_{n=3}^{\infty} \frac{\sqrt{n}}{(n-1)^{3/2}} \geq \sum_{n=3}^{\infty} \frac{1}{n} = \infty \Rightarrow \sum_{n=3}^{\infty} \frac{\sqrt{n}}{(n-1)^{3/2}} = \infty$$

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$$\sum_{n=-3}^{\infty} \frac{1}{n^2-2n+7}$$

n ranges over $-3, -2, -1, 0, 1, 2, 3, 4, 5, 6 \ldots\ldots$

Intuition: $\frac{1}{n^2-2n+7}$ is like $\frac{1}{n^2}$ when n is big.

Getting a rigorous proof using the direct comparison test:

For simplicity, restrict to $n \geq 0$.
The terms for $n = -3, -2, -1$ don't change wether the series diverges

$$\cancel{n^2 - 2n + 7 \geq n^2 + 7 \Rightarrow \frac{1}{n^2-2n+7} \le \frac{1}{n+7}}$$

$$n^2 - 2n + 7 = \left(n - \frac{2}{2}\right)^2 + 7 = (n-1)^2 - 1 + 7 = (n-1)^2 + 6$$

$$x^2 + px = \left(x + \frac{p}{2}\right)^2 - \left(\frac{p}{2}\right)^2$$

Continues $\Longrightarrow$

Getting a rigorous proof using the direct comparison test:

Idea: Show $\frac{1}{2}n^2 < n^2 - 2n + 7$ for all n big enough

This will imply $\frac{2}{n^2} > \frac{1}{n^2-2n+7} > 0$,

$\infty > 2 \sum_n \frac{1}{n^2} = \frac{2}{n^2} > \sum_n \frac{1}{n^2-2n+7} > 0$,

so $\sum_{n=-3}^{\infty} \frac{1}{n^2-2n+7}$ converges.

$\frac{1}{2}n^2 < n^2 - 2n + 7 \Leftrightarrow 0 < \frac{1}{2}n^2 - 2n + 7 \Leftrightarrow 0 < n^2 - 4n + 14 = (n-2)^2 + 10$

$0 < 10 \leq (n-2)^2 + 10 \Rightarrow 0 < \underbrace{(n-2)^2}_{\geq 0} + 10$

We wanted $\frac{1}{n^2} > \frac{1}{n^2-2n+7}$, which is false.

Instead we proved $\frac{2}{n^2} > \frac{1}{n^2-2n+7}$

$$\sum_{n=1}^{\infty} \frac{\ln(n)}{n} = \frac{\ln 1}{1} + \frac{\ln 2}{2} + \frac{\ln 3}{3} + \frac{\ln 4}{4} \ldots$$

$$= 0 + \frac{\ln 2}{2} + \underbrace{\frac{\ln 3}{3}}_{\geq \frac{1}{3}} + \underbrace{\frac{\ln 4}{4}}_{\geq \frac{1}{4}} + \underbrace{\frac{\ln 5}{5}}_{\geq \frac{1}{5}}$$

$3 > e \Rightarrow \ln 3 > \ln e = 1$

$$\sum_{n=1}^{\infty} \frac{\ln(n)}{n} = 0 + \frac{\ln 2}{2} + \sum_{n=3}^{\infty} \frac{\ln(n)}{n} \geq 0 + \frac{\ln 2}{2} + \underbrace{\sum_{n=3}^{\infty} \frac{1}{n}}_{\infty} = \infty$$