Today: 11.4: Limit comparison test

Monday: $\begin{cases}\text{Long HW10 DUE @ 5 PM} \\ 11.5\end{cases}$

- $\sum_{n=6}^{\infty} \frac{4\sqrt{n}}{n^3-5n+2}$ convergent or divergent?

Intuition: when n is big, $n^3 >> 5n$ and $n^3 >> 2$, so $\frac{4\sqrt{n}}{n^3-5n+2}$ is like $\frac{4\sqrt{n}}{n^3}$ and $\sum_{n=6}^{\infty} \frac{4\sqrt{n}}{n^3} = 4\sum_{n=6}^{\infty} \frac{1}{n^{5/2}}$ converges

$$\lim_{n\to\infty} \frac{4\sqrt{n}}{n^3-5n+2} = \lim_{n\to\infty} \frac{4\sqrt{n}/n^3}{(n^3-5n+2)/n^3} = \lim_{n\to\infty} \frac{4/n^{5/2}}{1-5/n^2+2/n^3} = \frac{0}{1-0+0} = 0 \Rightarrow \text{inconclusive}$$

Remember $\lim_{n\to\infty} \frac{1}{n} = 0$ & $\lim_{n\to\infty} \frac{1}{n^2} = 0$, but $\sum_{n=1}^{\infty} \frac{1}{n}$ diverges & $\sum_{n=1}^{\infty} \frac{1}{n^2}$ converges

$$\boxed{\sum_{n=1}^{\infty} \frac{1}{n^p} \text{ is } \begin{cases}\text{convergent}, & p>1 \\ \text{divergent}, & p\le 1\end{cases}}$$

From Intuition to Proof:

$a_n = \frac{4\sqrt{n}}{n^3-5n+2}$ $\quad b_n = \frac{4\sqrt{n}}{n^3}$

$$\lim_{n\to\infty} \frac{a_n}{b_n} = \frac{4\sqrt{n}}{n^3-5n+2} \cdot \frac{n^3}{4\sqrt{n}} = \frac{n^3}{n^3-5n+2} = \lim_{n\to\infty} \frac{n^3/n^3}{(n^3-5n+2)/n^3} = \lim_{n\to\infty} \frac{1}{1-5/n^2+2/n^3}$$

$$\Leftarrow 1 = \frac{1}{1-0+0}$$

There is some H such that H is big enough that for all $n \ge H$, $.99 \le \frac{a_n}{b_n} \le 1.01$

$$0 \le \sum_{n=H}^{\infty} a_n \le \sum_{n=H}^{\infty} 1.01 b_n = 1.01 \underbrace{\sum_{n=H}^{\infty} b_n}_{\text{convergent}} \Rightarrow \sum_{n=H}^{\infty} a_n \text{ converges too}$$

$$\Rightarrow \sum_{n=6}^{\infty} a_n = a_6 + a_7 + a_8 + \dots + a_{H-2} + a_{H-1} + \sum_{n=H}^{\infty} a_n \text{ is convergent.}$$

Suppose you have $a_0, a_1, a_2, a_3, \ldots \geq 0$, and $b_0, b_1, b_2, \ldots \geq 0$.
Here's what happens

If we have	$\sum_{n=0}^{\infty} a_n < \infty$	$\sum_{n=0}^{\infty} b_n < \infty$	$\sum_{n=0}^{\infty} a_n = \infty$	$\sum_{n=0}^{\infty} b_n = \infty$
$\lim_{n\to\infty} \frac{a_n}{b_n} = c$ and $0 < c < \infty$	$\sum_{n=0}^{\infty} b_n < \infty$ too	$\sum_{n=0}^{\infty} a_n < \infty$ too	$\sum_{n=0}^{\infty} b_n = \infty$ too	$\sum_{n=0}^{\infty} a_n = \infty$ too
$\lim_{n\to\infty} \frac{a_n}{b_n} = 0$	inconclusive	$\sum_{n=0}^{\infty} a_n < \infty$ too	$\sum_{n=0}^{\infty} b_n = \infty$ too	inconclusive
$\lim_{n\to\infty} \frac{a_n}{b_n} = \infty$	$\sum_{n=0}^{\infty} b_n < \infty$ too	inconclusive	inconclusive	$\sum_{n=0}^{\infty} a_n = \infty$ too

$a_n = \frac{1}{n^3}$, $b_n = \frac{1}{n^2}$, $\lim_{n\to\infty} \frac{a_n}{b_n} = \lim_{n\to\infty} \frac{1/n^3}{1/n^2} = \lim_{n\to\infty} \frac{1}{n} = 0$

$\sum_{n=1}^{\infty} \frac{1}{n^3} < \infty$, $\sum_{n=1}^{\infty} \frac{1}{n^2} < \infty$

$a_n = \frac{1}{n^{3/2}}$ $b_n = \frac{1}{n^{1/2}}$, $\lim_{n\to\infty} \frac{a_n}{b_n} = \lim_{n\to\infty} \frac{1/n^{3/2}}{1/n^{1/2}} = \lim_{n\to\infty} \frac{1}{n} = 0$

$\sum_{n=1}^{\infty} \frac{1}{n^{3/2}} < \infty$, $\sum_{n=1}^{\infty} \frac{1}{n^{1/2}} = \infty$

Remember $\sum_{n=1}^{\infty} \frac{1}{n^p}$ is $\begin{cases} \text{convergent if } p > 1 \\ \text{divergent if } p \leq 1 \end{cases}$
p-series

$\sum_{n=0}^{\infty} r^n = \begin{cases} \frac{1}{1-r} ; -1 < r < 1 \\ \text{divergent o/w} \end{cases}$
Geometric series

$$\sum_{n=2}^{\infty} \frac{n^5}{3^n} = \frac{2^5}{3^2} + \frac{3^5}{3^3} + \frac{4^5}{3^4} + \frac{5^5}{3^5} + \frac{6^5}{3^6} + \frac{7^5}{3^7} + \frac{8^5}{3^8} + \frac{9^5}{3^9} + \ldots \qquad n = 20$$

$\approx 3.55 + 9 + 12.64 + 12.86 + 10.67 + 7.68 + 4.99 + 3 + 1.69 + 0.91 \ldots .000092$

$a_n = \frac{n^5}{3^n}$ $b_n = \frac{1}{2^n}$ I claim $\lim_{n\to\infty} \frac{a_n}{b_n} = 0$

Since $\sum_{n=0}^{\infty} \frac{1}{2^n} = \sum_{n=0}^{\infty} \left(\frac{1}{2}\right)^n = \frac{1}{1-\frac{1}{2}} < \infty$, $\sum_{n=2}^{\infty} a_n < \infty$ too.

$$\lim_{n\to\infty} \frac{a_n}{b_n} = \lim_{n\to\infty} \frac{n^5}{3^n} \cdot \frac{2^n}{1} = \lim_{n\to\infty} \frac{n^5 \to \infty}{1.5^n \to \infty}$$

$$= \lim_{x\to\infty} \frac{x^5 \to \infty}{1.5^x \to \infty} = \lim_{x\to\infty} \frac{(x^5)'}{(1.5^x)'} = \lim_{x\to\infty} \frac{5x^4 \to \infty}{1.5^x \ln 1.5 \to \infty}$$

↳ x any real

Use L'Hospital's rule 4 more times...

$$= \lim_{x\to\infty} \frac{5 \cdot 4 \cdot 3 \cdot 2 \cdot 1}{1.5^x \cdot \underbrace{\ln(1.5)\ln(1.5)\ldots\ln(1.5)}_{5 \text{ times}}}$$

$$= \frac{120}{\ln^5 1.5} \lim_{x\to\infty} \frac{1}{1.5^x} = 0$$