Today: 11.10 continued. 04/28/2010

$$e^x = \sum_{n=0}^{\infty} \frac{x^n}{n!}$$

$$e = e^1 = \sum_{n=0}^{\infty} \frac{1^n}{n!} = \sum_{n=0}^{\infty} \frac{1}{n!}$$

$$f(x) = \sum_{n=0}^{k} \frac{(x-a)^n}{n!} + R_k(x)$$

$$e^1 = \sum_{n=0}^{k} \frac{1}{n!} + \underbrace{R_k(1)}_{\text{error}}$$

If $|f^{(n+1)}(t)| \le M$ for all t between x & a, then

$$|R_k(x)| \le \frac{M|x-a|^{k+1}}{(k+1)!}$$

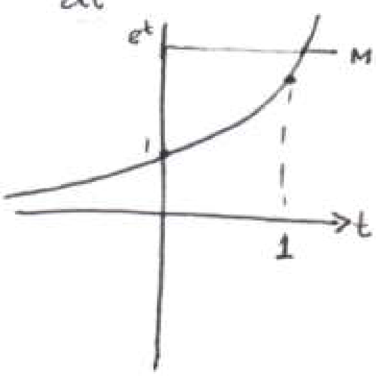
If $\left|\frac{d}{dt^{n+1}} e^t\right| \le M$ for all t between 1 & 0, then

$$|R_k(1)| \le \frac{M\,|1-0|^{k+1}}{(k+1)!} = \frac{M}{(k+1)!}$$

- First of all we can do a major notification

$$\frac{d^{n+1}}{dt^{n+1}} e^t = e^t$$

We're looking for M such that $|e^t| \le M$ for all t between 1 & 0.



Let's use $M = e$

$$|\text{error}| = |R_k(1)| \le \frac{e}{(k+1)!}$$

$$e = \frac{1}{0!} + \frac{1}{1!} + \frac{1}{2!} + \frac{1}{3!} + \underbrace{R_3(1)}_{\text{error}} \qquad |R_3(1)| \le \frac{e}{(3+1)!} = \frac{e}{24}$$

$$e = \frac{1}{1} + \frac{1}{1} + \frac{1}{2} + \frac{1}{6} + R_3(1) = 2 + \frac{2}{3} + \underbrace{R_3(1)}_{\text{error}}$$

$$2 + \frac{2}{3} - \frac{e}{24} \le e \le 2 + \frac{2}{3} + \frac{e}{24}$$

$$2 + \frac{2}{3} - \frac{e}{24} \le e$$

$$2 + \frac{2}{3} \le e + \frac{e}{24}$$

$$2 + \frac{2}{3} \le \frac{25}{24} e$$

$$\frac{24}{25}\left(2 + \frac{2}{3}\right) \le e$$

$$e \le 2 + \frac{2}{3} + \frac{e}{24}$$

$$e - \frac{e}{24} \le 2 + \frac{2}{3}$$

$$\frac{23}{24} e \le 2 + \frac{2}{3}$$

$$e \le \frac{23}{24}\left(2 + \frac{2}{3}\right)$$

$$\frac{24}{25}\left(2 + \frac{2}{3}\right) \le e \le \frac{24}{23}\left(2 + \frac{2}{3}\right)$$

$$\frac{64}{25} \le e \le \frac{64}{23}$$

2.56 $\qquad \approx 2.78261$

Now I know $e \leq \frac{64}{23} < \frac{69}{23} = 3$, so I'll use $M = 3$ to simplify future estimates.

$$e = \frac{1}{0!} + \frac{1}{1!} + \frac{1}{2!} + \ldots \frac{1}{10!} + R_{10}(1)$$

$$|R_{10}(1)| \leq \frac{M}{(10+1)!} = \frac{3}{(10+1)!} \approx \underbrace{7.5156 \times 10^{-8}}_{.000000075156}$$

$$e \approx \frac{1}{0!} + \frac{1}{1!} + \ldots \frac{1}{10!} = \frac{9864101}{3628800} \approx \underbrace{2.71828180}_{\text{Correct}}11464$$

EXAMPLE #2

$f(x) = \sqrt{x}$

$a = 1 \qquad x = 1.3$

Estimate $\sqrt{1.3}$

$f(1) = \sqrt{1} = 1$ $\qquad f'(x) = \frac{1}{2\sqrt{x}} = \frac{1}{2}x^{-1/2}$

$f'(1) = \frac{1}{2\sqrt{1}} = \frac{1}{2}$ $\qquad f''(x) = \frac{1}{2}\left(-\frac{1}{2}\right)x^{-3/2}$

$f''(1) = \frac{1}{2}\left(-\frac{1}{2}\right)(1) = -\frac{1}{4}$ $\qquad f^{(3)}(x) = \frac{1}{2}\left(-\frac{1}{2}\right)\left(-\frac{3}{2}\right)x^{-5/2}$

$f^{(3)}(1) = \frac{1}{2}\left(-\frac{1}{2}\right)\left(-\frac{3}{2}\right)(1) = \frac{3}{8}$ $\qquad f^{(4)} = \frac{1}{2}\left(-\frac{1}{2}\right)\left(-\frac{3}{2}\right)\left(-\frac{5}{2}\right)x^{-7/2}$

$$f(1.3) = f(1) + f'(1)(1.3-1) + \frac{f''(1)}{2!}(1.3-1)^2 + \frac{f'''(1)}{3!}(1.3-1)^3 + \underbrace{R_3(1.3)}_{\text{error.}}$$

- If $|f^{(4)}(t)| \leq M$ for all t between 1 & 1.3, then $|R_3(1.3)| \leq M\frac{(1.3-1)^4}{4!}$

$$f^{(4)}(t) = \frac{-15}{16}t^{-7/2} = -\frac{15}{16t^{7/2}}$$

$$1 \leq t \leq 1.3 < 1.44 = 1.2^2$$

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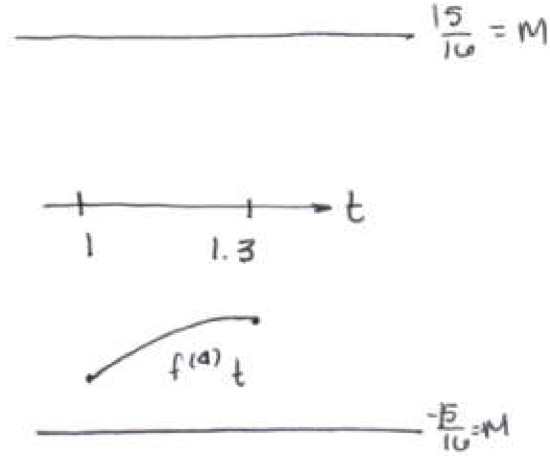
$$1 = 1^{7/2} \le t^{7/2} < (1.2^2)^{7/2} = 1.2^7 = 3.5831808 < 4$$

$$\frac{1}{1} \ge \frac{1}{t^{7/2}} \ge \frac{1}{4}$$

$$\frac{15}{16} \ge \frac{15}{16t^{7/2}} \ge \frac{15}{64}$$

$$-M = \frac{-15}{16} \le -\frac{15}{16t^{7/2}} \le -\frac{15}{64} < \frac{15}{16} = M$$



$$|R_3(1.3)| \le M\,\frac{(1.3-1)^4}{4!} = \frac{(15/16)(.3)^4}{4!} = .000315\ldots$$

$$* \; f(1.3) = f(1) + f'(1)(1.3-1) + \frac{f''(1)}{2!}(1.3-1)^2 + \frac{f'''(1)}{3!}(1.3-1)^3 + \underbrace{R_3(1.3)}_{\text{error.}}$$

$$= 1 + \frac{1}{2}(.3) - \frac{1}{8}(.09) + \frac{1}{16}(.027) + R_3(1.3)$$

$$\sqrt{1.3} \approx \underbrace{1 + \frac{.3}{2} - \frac{.09}{8} + \frac{.027}{16}}$$

$$\approx 1.4044$$

$$|\text{error}| \le .000315$$