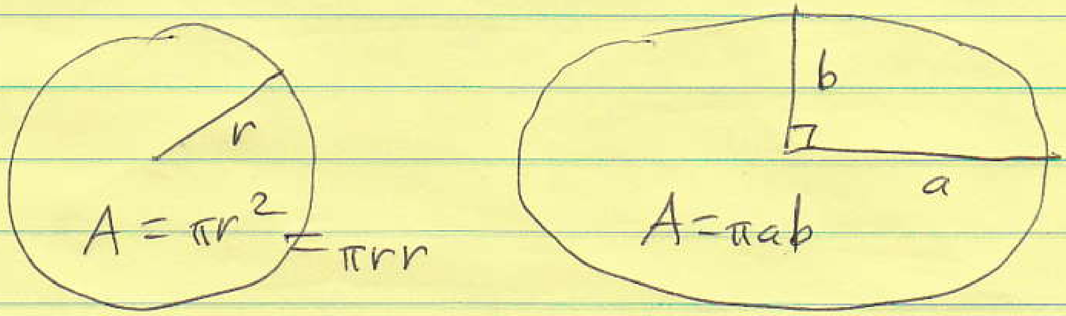
Today: Powers of secant & tangent (also in 7.5 of Keisler)

First, HW #1: Parametrize the ellipse $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$

as $x =$ ________
$y =$ ________ and
$0 \le t \le 2\pi$

prove that this ellipse has area $\pi \cdot a \cdot b$ (assuming $a, b > 0$).



$$d(\tan x) = \sec^2 x \, dx$$

$$d(\sec x) = \sec x \tan x \, dx$$

$$\tan x + c = \int \sec^2 x \, dx$$

$$\sec x + c = \int \sec x \tan x \, dx$$

$$\tan x = \frac{\sin x}{\cos x} \qquad \sec x = \frac{1}{\cos x}$$

$$\sec \frac{\pi}{3} = \frac{1}{\cos \pi/3} = \frac{1}{1/2} = 2$$

$$\int \tan x \, dx = \int \frac{\sin x}{\cos x} dx = \int \frac{\sin x \, dx}{\cos x}$$

$u = \cos x \qquad du = -\sin x \, dx$

$\qquad\qquad\qquad -du = \sin x \, dx$

$$\int \tan x \, dx = \int \frac{-du}{u} = -\ln|u| + c$$

$$= -\ln|\cos x| + c$$

$$= \ln\left|\frac{1}{\cos x}\right| + c$$

$$= \ln|\sec x| + c$$

$$\int \sec x \, dx = \int \sec x \frac{\sec x + \tan x}{\sec x + \tan x} dx$$

$$= \int \frac{(\sec^2 x + \sec x \tan x)\, dx}{\sec x + \tan x}$$

$$d(\sec x) = \sec x \tan x \, dx$$
$$d(\tan x) = \sec^2 x \, dx$$

$$= \int \frac{d(\tan x + \sec x)}{\tan x + \sec x} = \int \frac{du}{u} = \ln|u| + c$$

$$u = \tan x + \sec x$$

$$\int \sec x \, dx = \ln|\sec x + \tan x| + c$$

$$\int \sec^3 x \, dx = \frac{1}{2} \sec x \tan x + \frac{1}{2} \ln|\sec x + \tan x| + c$$

Keep $\int \sec x \, dx$, $\int \sec^3 x \, dx$,

and $\int \tan x \, dx$ in your notes,

as well as the basics

like $\int \sec^2 x \, dx$

$\int \tan^3 x \, dx = ?$

$\int \tan x \tan^2 x \, dx =$

$\frac{\cos^2 x}{\cos^2 x} + \frac{\sin^2 x}{\cos^2 x} = \frac{1}{\cos^2 x}$

$1 + \tan^2 x = \sec^2 x$

$\tan^2 x = \sec^2 x - 1$

$\int \tan x (\sec^2 x - 1) dx$

$\int \underbrace{\tan x}_{u} \underbrace{\sec^2 x \, dx}_{du} - \underbrace{\int \tan x \, dx}_{\ln|\sec x| + c}$

$\int u \, du = \frac{1}{2} u^2 + c$

$= \boxed{\frac{1}{2} \tan^2 x + \ln|\sec x| + c}$

$= \int \tan^3 x \, dx$

General rule: $\int \tan^{2m+1} x \, dx$

$= \int (\tan x)^{2m-1} (\sec^2 x - 1) dx$

$$= \int \overbrace{\tan^{2m-1} x}^{u^{2m-1}} \, \overbrace{\sec^2 x \, dx}^{du} \qquad u = \tan x$$

$$+ \int - \underbrace{\tan^{2m-1} x} \, dx$$

If $2m-1 \geq 1$, then repeat:

$$\tan^{2m-1} x = \tan^{2m-3} x \, (\sec^2 x - 1)$$

$$\int \tan^5 x \, dx = \int \tan^3 x \, \tan^2 x \, dx$$

$$= \int \tan^3 x \, (\sec^2 x - 1) \, dx$$

$$= \underbrace{\int \underbrace{\tan^3 x}_{u^3} \, \underbrace{\sec^2 x \, dx}_{du}}_{u = \tan x} - \underbrace{\int \tan^3 x \, dx}_{\underbrace{\int \tan x (\sec^2 x - 1) \, dx}}$$

$$\int u^3 \, du = \frac{1}{4} u^4 + c$$

$$= \frac{1}{4} \tan^4 x + c$$

We did this one already:

$$\frac{1}{2} \tan^2 x + \ln|\sec x| + c$$

$$\int \tan^5 x \, dx = \frac{1}{4} \tan^4 x - \frac{1}{2} \tan^2 x - \ln|\sec x| + c$$

That's how you handle odd powers of tangent.

$$\int \tan^5 x \sec^3 x \, dx$$

$$= \int \underbrace{\underbrace{\underbrace{\tan^4 x}_{(\tan^2 x)^2}}_{(\sec^2 x - 1)^2}}_{(u^2-1)^2} \; \underbrace{\sec^2 x}_{u^2} \; \underbrace{\sec x \tan x \, dx}_{du}$$

$u = \sec x$

$$= \int (u^2 - 1)^2 u^2 \, du$$

$$= \int (u^4 - 2u^2 + 1) u^2 \, du$$

$$= \int (u^6 - 2u^4 + u^2) \, du$$

$$= \frac{u^7}{7} - \frac{2u^5}{5} + \frac{u^3}{3} + c$$

$$= \frac{\sec^7 x}{7} - \frac{2}{5} \sec^5 x + \frac{\sec^3 x}{3} + c$$

Strategy: Try to pull out a factor of $\sec^2 x \, dx$ or $\sec x \tan x \, dx$ as your du.

HW #2 $\displaystyle\int_0^{\pi/6} \tan^3 x \sec^5 x \, dx = ?$

$$\int \sec^4 x \, dx = \int \underbrace{\underbrace{\sec^2 x}_{1+\tan^2 x}}_{1+u^2} \underbrace{\sec^2 x \, dx}_{du}$$

$u = \tan x$

$$\int (1+u^2) \, du$$

$$= u + \frac{u^3}{3} + c = \tan x + \frac{1}{3}\tan^3 x + c$$

HW #3 $\displaystyle\int \tan^5 x \sec^4 x \, dx = ?$

Tricky cases: $\displaystyle\int (\tan^{\text{even}} x)(\sec^{\text{odd}} x) \, dx$

like $\sec x$, $\sec^3 x$

HW #4 $\int_{-\pi/4}^{0} \tan^5(x+\pi/4)\sec(x+\pi/4)\,dx$