HW #1 Find the length of the parabolic arc $\begin{cases} x = 2t^2 \\ y = (t-1)^2 \\ -1 \le t \le 3. \end{cases}$

More uses of $1 + \tan^2\theta = \sec^2\theta$:

$$\int \frac{dx}{4+5x^2} = \int \frac{dx}{4(1+(5/4)x^2)}$$

$$= \frac{1}{4}\int \frac{dx}{1+(5/4)x^2} = \frac{1}{4}\int \frac{dx}{\underbrace{1+((\sqrt{5}/2)x)^2}_{\underbrace{1+\tan^2\theta}_{\sec^2\theta}}}$$

$$\frac{\sqrt{5}}{2}x = \tan\theta$$

$$\frac{\sqrt{5}}{2}\,dx = \sec^2\theta\,d\theta$$

$$dx = \frac{2}{\sqrt{5}}\sec^2\theta\,d\theta$$

$$\frac{1}{4}\int \frac{(2/\sqrt{5})\sec^2\theta\,d\theta}{\sec^2\theta}$$

$$\frac{1}{4}\int \frac{2}{\sqrt{5}}\,d\theta = \frac{1}{4}\cdot\frac{2}{\sqrt{5}}\,\theta + c$$

$$\arctan\left(\frac{\sqrt{5}}{2}x\right) = \theta$$

assuming $-\frac{\pi}{2} < \theta < \frac{\pi}{2}$

$$\int \frac{dx}{4+5x^2} = \cancel{\arctan\left(\frac{\sqrt{5}}{2}x\right)}$$

$$\frac{1}{4} \cdot \frac{2}{\sqrt{5}} \arctan\left(\frac{\sqrt{5}}{2}x\right) + c$$

$$= \frac{1}{2\sqrt{5}} \arctan\left(\frac{\sqrt{5}}{2}x\right) + c$$

In other terms:

$$\left(\frac{1}{2\sqrt{5}} \arctan\left(\frac{\sqrt{5}}{2}x\right) + c\right)' = \frac{1}{4+5x^2}$$

Recall $(\arctan x)' = \dfrac{1}{1+x^2}$

$$\boxed{\text{HW \#2} \quad \int_{-1}^{1} \frac{dx}{3+x^2} = ?}$$

$$I = \int \frac{dx}{\sqrt{3+4x+5x^2}} = \int \frac{dx}{\sqrt{5}\sqrt{\frac{3}{5}+\frac{4}{5}x+x^2}}$$

Completing the square:

$$x^2+px = \left(x+\frac{p}{2}\right)^2 - \left(\frac{p}{2}\right)^2$$

$$x^2+\frac{4}{5}x = \left(x+\frac{2}{5}\right)^2 - \left(\frac{2}{5}\right)^2$$

$$x^2 + \frac{4}{5}x + \frac{3}{5} = \left(x + \frac{2}{5}\right)^2 \underbrace{- \frac{4}{25} + \frac{3}{5}}_{+11/25}$$

$$I = \int \frac{dx}{\sqrt{5}\sqrt{\left(x+\frac{2}{5}\right)^2 + 11/25}}$$

$$= \int \frac{dx}{\sqrt{5}\sqrt{\frac{11}{25}}\sqrt{\frac{25}{11}\left(x+\frac{2}{5}\right)^2 + 1}}$$

$$I = \int \frac{dx}{\sqrt{5}\sqrt{\frac{11}{25}}\underbrace{\sqrt{\left(\sqrt{\frac{25}{11}}\left(x+\frac{2}{5}\right)\right)^2 + 1}}}$$

$$\sec\theta = \sqrt{\sec^2\theta} = \sqrt{\tan^2\theta + 1}$$

assuming $-\pi/2 < \theta < \pi/2$

$$\sqrt{\frac{25}{11}}\left(x + \frac{2}{5}\right) = \tan\theta$$

$$\sqrt{\frac{25}{11}}\left(dx + 0\right) = \sec^2\theta \, d\theta$$

$$dx = \sqrt{\frac{11}{25}}\sec^2\theta \, d\theta$$

$$I = \int \frac{\cancel{\sqrt{\frac{11}{25}}}\sec^2\theta \, d\theta}{\sqrt{5}\cancel{\sqrt{\frac{11}{25}}}\sec\theta} = \int \frac{1}{\sqrt{5}}\sec\theta \, d\theta$$

$$I = \frac{1}{\sqrt{5}} \ln \left| \sec\theta + \underbrace{\tan\theta}_{\sqrt{\frac{25}{11}}\left(x+\frac{2}{5}\right)} \right| + c$$

$$\sec\theta = \sqrt{1+\tan^2\theta} = \sqrt{1+\left(\frac{25}{11}\right)\left(x+\frac{2}{5}\right)^2}$$

$$\int \frac{dx}{\sqrt{3+4x+5x^2}} =$$

$$= \frac{1}{\sqrt{5}} \ln \left| \sqrt{1+\frac{25}{11}\left(x+\frac{2}{5}\right)^2} + \sqrt{\frac{25}{11}}\left(x+\frac{2}{5}\right) \right| + c$$

HW #3 $\displaystyle\int \frac{x\,dx}{\sqrt{2+x+2x^2}} = ?$

$$I = \int \frac{dx}{x^2\underbrace{(1+x^2)}^2} = \int \frac{\sec^2\theta\, d\theta}{\tan^2\theta\,(\sec^2\theta)^2}$$

$$1+x^2 = 1+\tan^2\theta = \sec^2\theta$$

Pick $x = \tan\theta$, $-\frac{\pi}{2} < \theta < \frac{\pi}{2}$

$$dx = \sec^2\theta\, d\theta$$

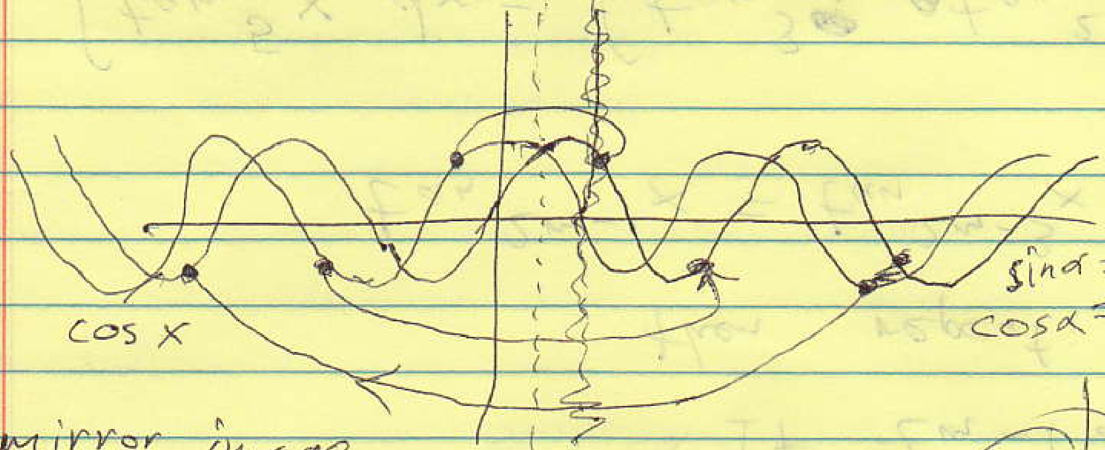
$$I = \int \frac{\sec^2\theta \, d\theta}{\tan^2\theta \sec^4\theta} = \int \frac{d\theta}{\tan^2\theta \sec^2\theta}$$

$$= \int \frac{d\theta}{\frac{\sin^2\theta}{\cos^2\theta} \cdot \frac{1}{\cos^2\theta}} = \int \frac{\cos^2\theta}{\sin^2\theta} \cdot \cos^2\theta \, d\theta$$

$$= \int \cot^2\theta \cos^2\theta \, d\theta$$

If you don't want to deal with $\cot\theta$, use $u = \pi/2 - \theta$ (same as $\theta = \pi/2 - u$)

$\cos(\pi/2 - \alpha) = \sin\alpha$
$\sin(\pi/2 - \alpha) = \cos\alpha$

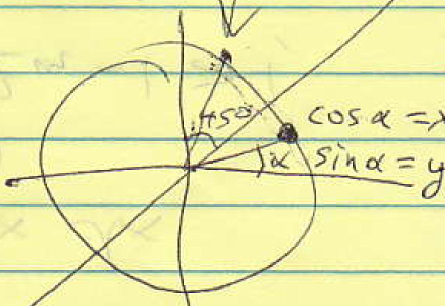


$\sin\alpha = \cos(\pi/2 - \alpha)$
$\cos\alpha = \sin(\pi/2 - \alpha)$

mirror image of $\cos x$ is $\sin x$

fixed

$\pi/2 - \pi/4 = \pi/4$



α

$\pi/2 - \alpha$

$$I = \int \cot^2\theta \cos^2\theta \, d\theta$$

$\theta = \pi/2 - u$

$\cot\theta = \cos\theta / \sin\theta = \sin u / \cos u$

$= \tan u$

$\cos\theta = \sin u$

$d\theta = 0 - du = -du$

$$I = \int \tan^2 u \sin^2 u \, (-du)$$

$$= \int \frac{\sin^2 u}{\cos^2 u} \sin^2 u \, (-du)$$

$$= \int \frac{1-\cos^2 u}{\cos^2 u} \sin^2 u \, (-du)$$

$\cos^2 u + \sin^2 u = 1$

$\sin^2 u = 1 - \cos^2 u$

$$\int \left(\frac{1}{\cos^2 u} - \frac{\cos^2 u}{\cos^2 u}\right) \sin^2 u \, (-du)$$

$$= \int \left(-\frac{\sin^2 u}{\cos^2 u} + \sin^2 u\right) du$$

$$= \int \left(-\tan^2 u + \sin^2 u\right) du$$

$$= -\int \tan^2 u \, du + \int \sin^2 u \, du$$

$$I = -\int (\sec^2 u - 1)\,du + \int \frac{1 - \cos 2u}{2}\,du$$

$$= -(\tan u - u) + \int \frac{1 - \cos v}{2}\,\frac{dv}{2}$$

$v = 2u$

$dv = 2\,du$

$dv/2 = du$

$$I = -\tan u + u + \frac{1}{2}\left(\frac{1}{2}\right)(v - \sin v) + c$$

$$I = -\overbrace{\tan(\pi/2 - \theta)}^{\cot\theta} + \pi/2 - \theta + \frac{1}{4}(2u - \sin 2u) + c$$

$$I = -\cot\theta + \pi/2 - \theta + \frac{2(\pi/2 - \theta)}{4} - \frac{1}{4}\sin(2(\pi/2 - \theta)) + c$$

Remember $x = \tan\theta$

$$I = -\underbrace{\frac{1}{\tan\theta}}_{1/x} + \pi/2 - \arctan x + \frac{\pi}{4} - \frac{1}{2}\arctan x - \frac{1}{4}\sin(\pi - 2\theta) + c$$

$$\sin(\alpha \pm \beta) = \sin\alpha \cos\beta \pm \sin\beta \cos\alpha$$

$$\sin(\pi - 2\theta) = \underbrace{\sin\pi}_{0} \cos 2\theta - \sin 2\theta \underbrace{\cos\pi}_{-1}$$

$$= \sin 2\theta = \sin(\theta + \theta)$$

$$= \sin\theta\cos\theta + \sin\theta\cos\theta$$

$$= 2\sin\theta\cos\theta$$

$x = \tan\theta$; $\sin\theta = ?$; $\cos\theta = ?$

$$1 + x^2 = 1 + \tan^2\theta = \sec^2\theta = \frac{1}{\cos^2\theta}$$

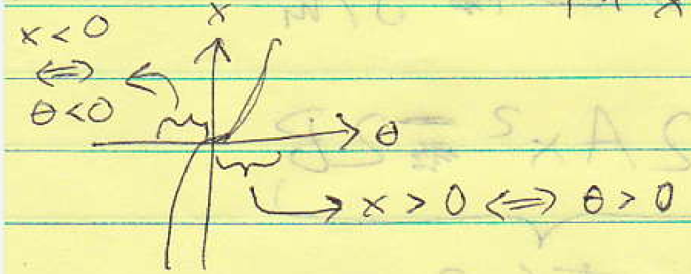
$$\Rightarrow \frac{1}{1+x^2} = \cos^2\theta \Rightarrow \frac{1}{\sqrt{1+x^2}} = \cos\theta$$

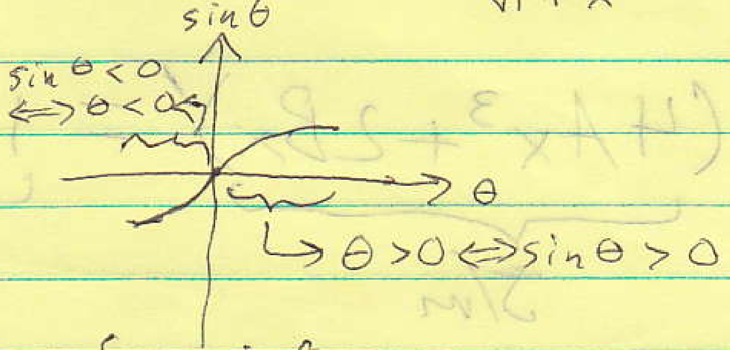
$\cos\theta > 0$ if $-\frac{\pi}{2} < \theta < \frac{\pi}{2}$

$$1 = \cos^2\theta + \sin^2\theta$$

$$\sin^2\theta = 1 - \cos^2\theta = 1 - \frac{1}{1+x^2} = \frac{1+x^2}{1+x^2} - \frac{1}{1+x^2}$$

$$\sin^2\theta = \frac{x^2}{1+x^2} \Rightarrow \sin\theta = \frac{\pm x}{\sqrt{1+x^2}}$$





$$\begin{cases} x = \tan\theta \\ -\frac{\pi}{2} < \theta < \frac{\pi}{2} \end{cases}$$

$$\begin{cases} \sin\theta \\ -\frac{\pi}{2} < \theta < \frac{\pi}{2} \end{cases}$$

So, $\begin{cases} \sin\theta > 0 \iff \theta > 0 \iff x > 0 \\ \sin\theta < 0 \iff \theta < 0 \iff x < 0. \end{cases}$

(assuming $-\pi/2 < \theta < \pi/2$)

Therefore, $\sin\theta = \dfrac{x}{\sqrt{1+x^2}}$

$$\int \frac{dx}{x^2(1+x^2)^2} = I = \boxed{-\frac{1}{x} - \frac{3}{2}\arctan x - \frac{x}{2(1+x^2)} + c}$$

$$\cos\theta \sin\theta = \frac{1}{\sqrt{1+x^2}}\,\frac{x}{\sqrt{1+x^2}} = \frac{x}{1+x^2}$$

$$\frac{1}{4}\underbrace{(2\cos\ \sin\theta)}_{\sin(\pi-2\theta)} = \frac{x}{2(1+x^2)}$$

combine all constants added together