Today: sine substitution.
(also in 7.6 of Keisler's book).

Test on Thursday (notes OK; calculators not OK)

(Recall $\sec^2\theta = 1+\tan^2\theta$.)

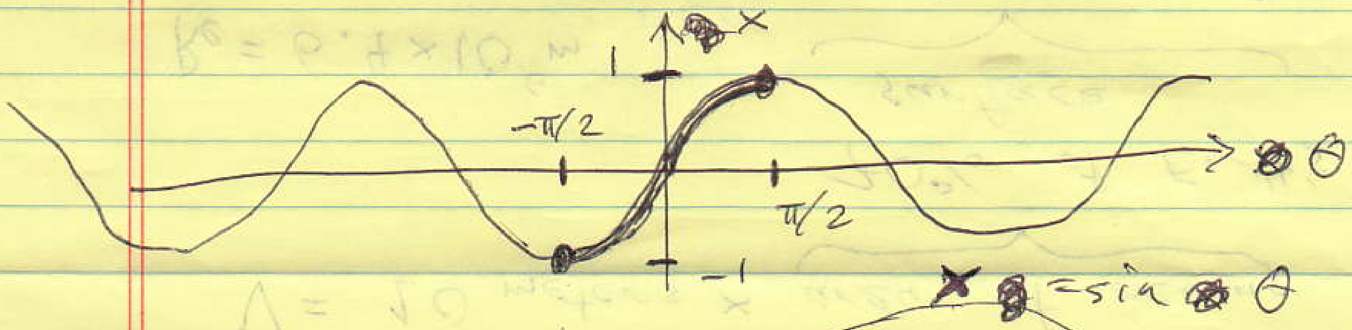
sine substitution takes advantage of

$$\cos^2\theta = 1-\sin^2\theta$$

If $-\pi/2 \le \theta \le \pi/2$, then
$\cos\theta \ge 0$, so $\cos\theta = \sqrt{1-\sin^2\theta}$.

$\sin^{-1} x = \arcsin x$ is the θ satisfying
$-\pi/2 \le \theta \le \pi/2$ and $\sin\theta = x$.

x must be in $[-1, 1]$.



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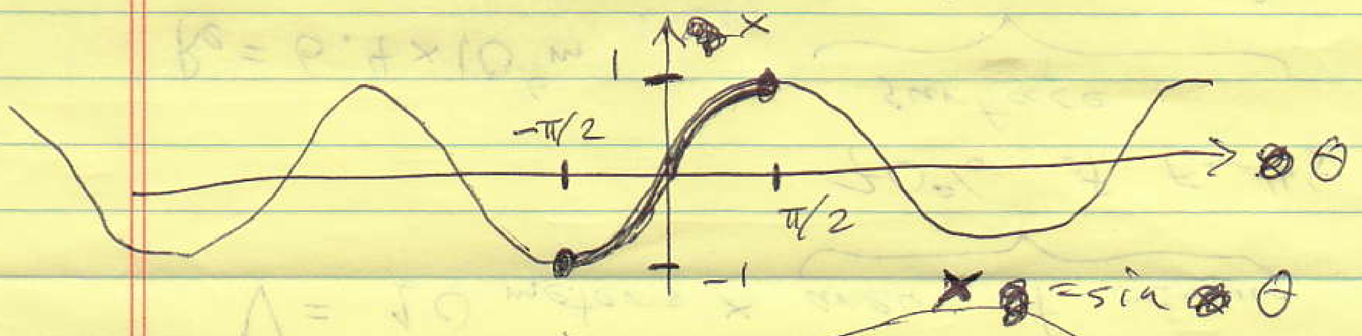
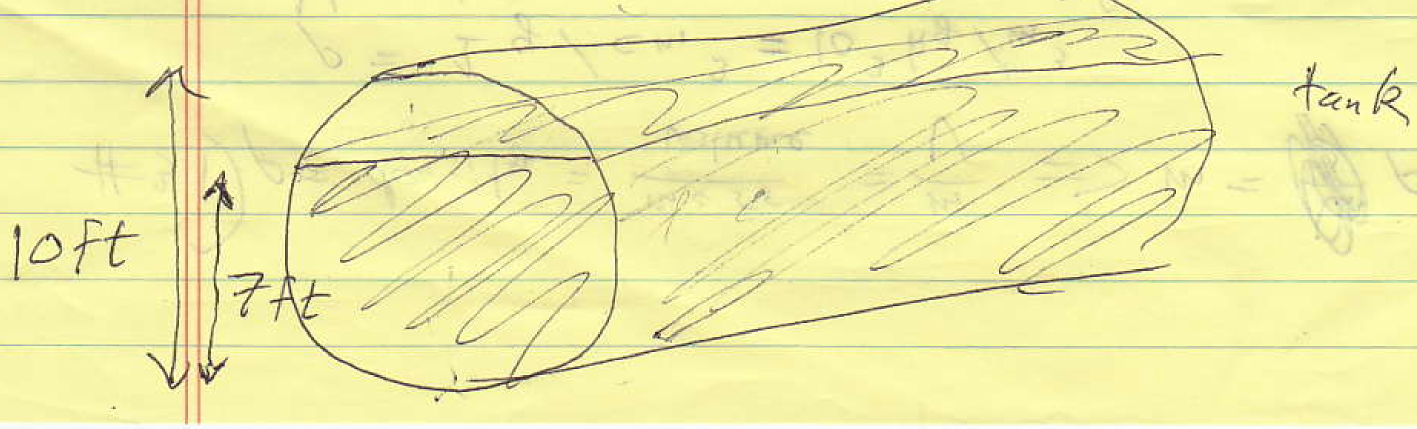
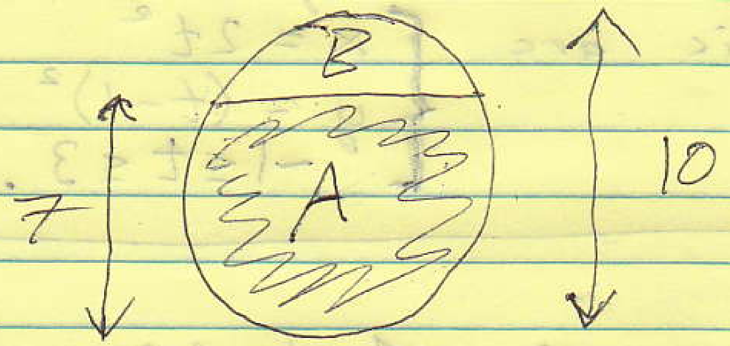




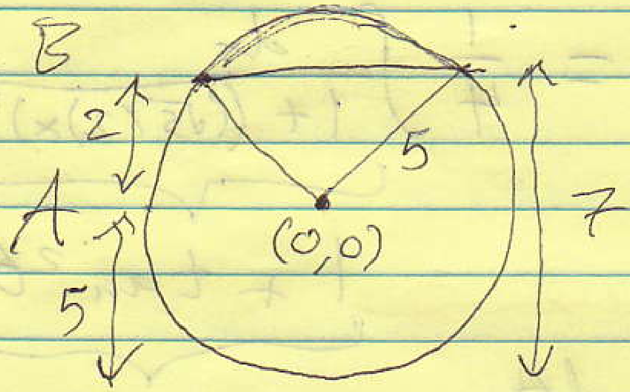
What fraction of the tank is full?



$$\text{Answer} = \frac{A}{A+B}$$

$$A+B = \pi \cdot 5^2$$

5 = radius



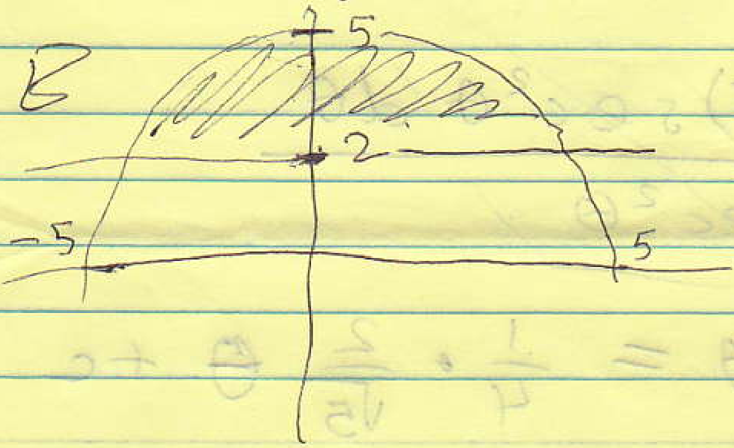
If we find B,

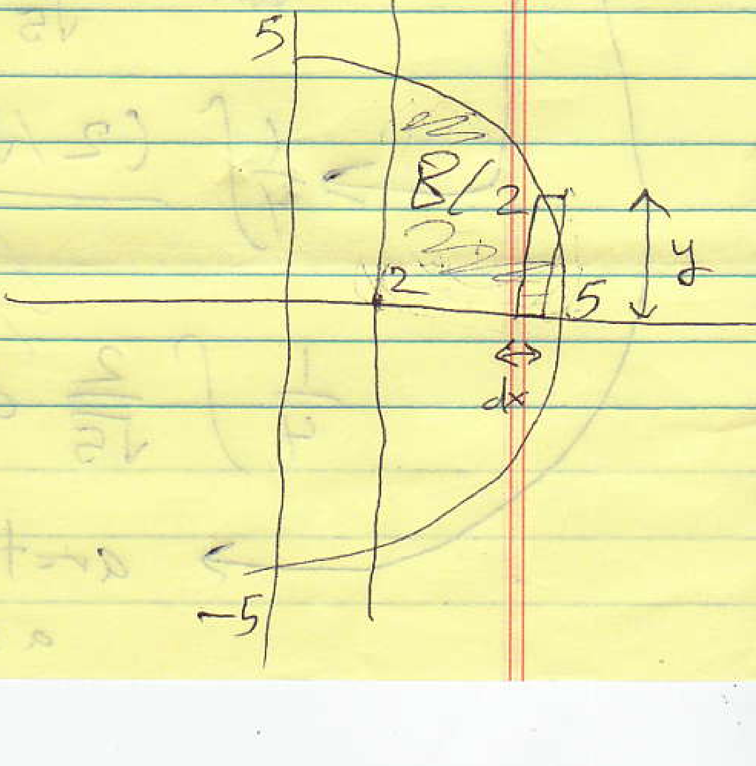
then $A = \pi \cdot 5^2 - B$,

so we know A too.

circle:

$$x^2+y^2=5^2$$





$$\frac{B}{2} = \int_{x=2}^{x=5} y\,dx = \int_2^5 \underbrace{\sqrt{25-x^2}}_{\underbrace{1-\sin^2\theta}_{\cos\theta}}\,dx$$

$$x^2 + y^2 = 5^2$$

$$y^2 = 5^2 - x^2$$

$$y = \sqrt{25-x^2}$$

$$\frac{B}{2} = \int_2^5 \sqrt{25}\,\sqrt{\underbrace{1 - x^2/25}_{1-\sin^2\theta}}\,dx$$

$$\sin^2\theta = x^2/25 \Rightarrow \sin\theta = \pm x/5$$

Pick $\sin\theta = x/5$ & $-\frac{\pi}{2} \le \theta \le \pi/2$

$$\cos\theta\,d\theta = dx/5$$

$$5\cos\theta\,d\theta = dx$$

$$\sqrt{1-x^2/25} = \sqrt{1-\sin^2\theta} = \cos\theta$$

B

$$\theta = \arcsin(x/5)$$

$$x = 5 \Rightarrow \theta = \arcsin(5/5)$$

$$= \arcsin(1) = \pi/2$$

$$x = 2 \Rightarrow \theta = \arcsin(2/5)$$

$$\approx 23.57° \approx 0.41$$

$$\frac{B}{2} = \int_2^5 \sqrt{25}\sqrt{1-x^2/25}\,dx$$

$$= \int_{\arcsin(2/5)}^{\pi/2} (5)(\cos\theta)(5\cos\theta\, d\theta)$$

$$= 25\int_{\arcsin(2/5)}^{\pi/2} \cos^2\theta\, d\theta$$

$$= 25\int_{\arcsin(2/5)}^{\pi/2} \frac{1+\cos 2\theta}{2}\, d\theta$$

$u = 2\theta \qquad du = 2d\theta$

$du/2 = d\theta$

$\theta = \pi/2 \Rightarrow u = \pi$

$\theta = \arcsin(2/5) \Rightarrow u = 2\arcsin\left(\frac{2}{5}\right)$

$$\frac{B}{2} = 25\int_{2\arcsin(2/5)}^{\pi} \frac{1+\cos u}{2}\,\frac{du}{2}$$

$$\frac{B}{2} = 25\left(\frac{u+\sin u}{2}\right)\left(\frac{1}{2}\right)\Bigg|_{2\arcsin(2/5)}^{\pi}$$

$$\frac{B}{2} = \frac{25}{4}\left[\left(\pi + \underbrace{\sin\pi}_{0}\right) - \left(2\arcsin\frac{2}{5} + \sin\left(2\arcsin\frac{2}{5}\right)\right)\right]$$

$$\approx 9.908$$

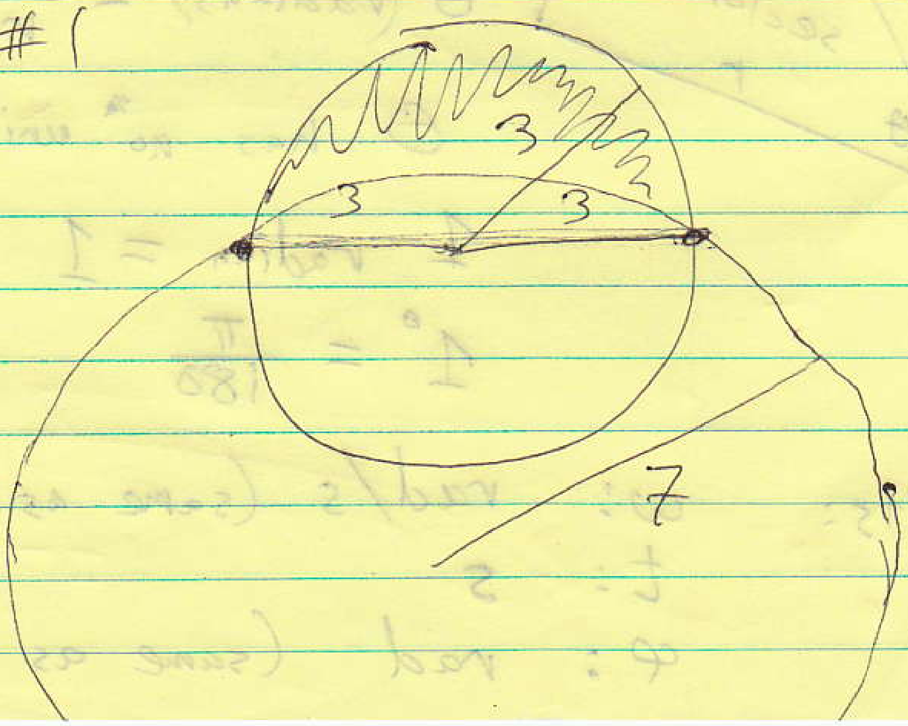
$$B \approx 19.82$$

$$A = \pi \cdot 5^2 - B \approx 58.72$$

$$\text{Answer} = \frac{A}{A+B} = \frac{A}{\pi \cdot 5^2} \approx 0.74768$$

$\Rightarrow$ tank about 75% full.

HW #1



Find the area of the crescent resulting from removing from a circle of radius 3, a circle of radius 7 that intersects the smaller circle at two opposite points.

HW #2 $\displaystyle\int \frac{dx}{\sqrt{4-x^2}} = ?$

HW #3 $\displaystyle\int_1^2 \frac{dx}{x\sqrt{8-x^2/3}} = ?$

HW #4 $\displaystyle\int \frac{dx}{\sqrt{2+15x-x^2}} = ?$

Since last test:

u-Substitution

$\int$(Powers of sine, tangent, cosine, secant)

Trigonometric substitutions

$$\int \frac{dx}{\underbrace{\sqrt{1-x^2}}_{\underbrace{1-\sin^2\theta}_{\cos\theta}}} = \int \frac{\cancel{\cos\theta}\, d\theta}{\cancel{\cos\theta}} = \int d\theta = \theta + c$$

$x = \sin\theta$ & $-\frac{\pi}{2} \le \theta \le \frac{\pi}{2}$

$dx = \cos\theta \, d\theta$

$\Rightarrow \theta = \arcsin x$

$$\int \frac{dx}{\sqrt{1-x^2}} = \arcsin x + c$$

$$\frac{1}{\sqrt{1-x^2}} = (\arcsin x)'$$

Similarly, $\frac{1}{1+x^2} = (\arctan x)'$

$$\int \frac{dx}{1+x^2} = \cdots = \arctan x + c$$

$\uparrow_{x = \tan\theta}$