Today: secant substitution

Tomorrow: review

Thursday: Test #4

Recall using $\sqrt{1+\tan^2\theta} = \sec\theta$

for integrals like $\int\sqrt{1+x^2}\,dx$

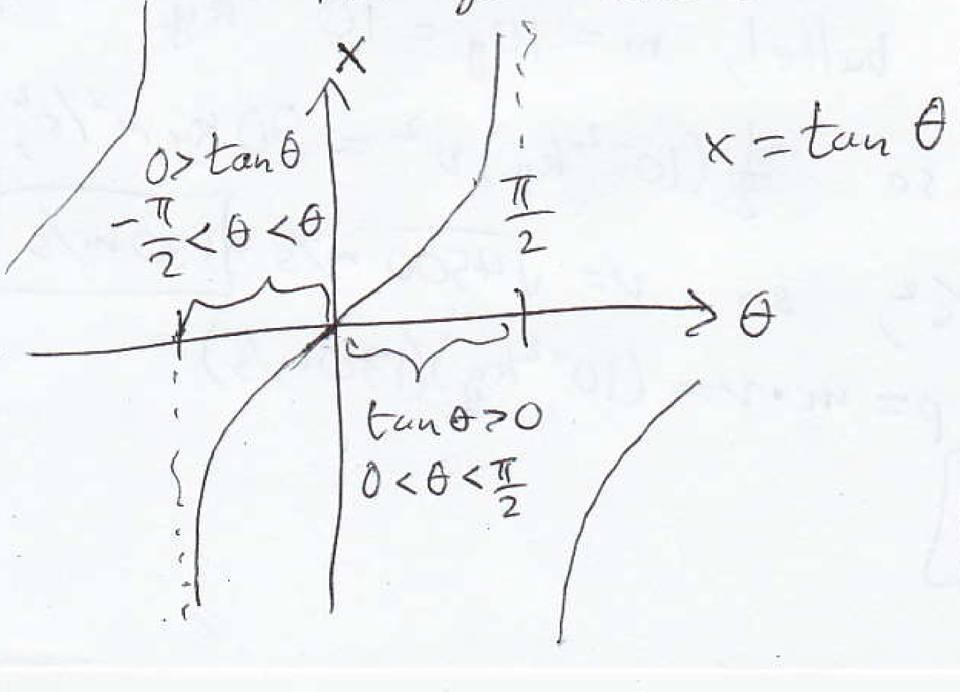
Then, using $\sqrt{1-\sin^2\theta} = \cos\theta$ for

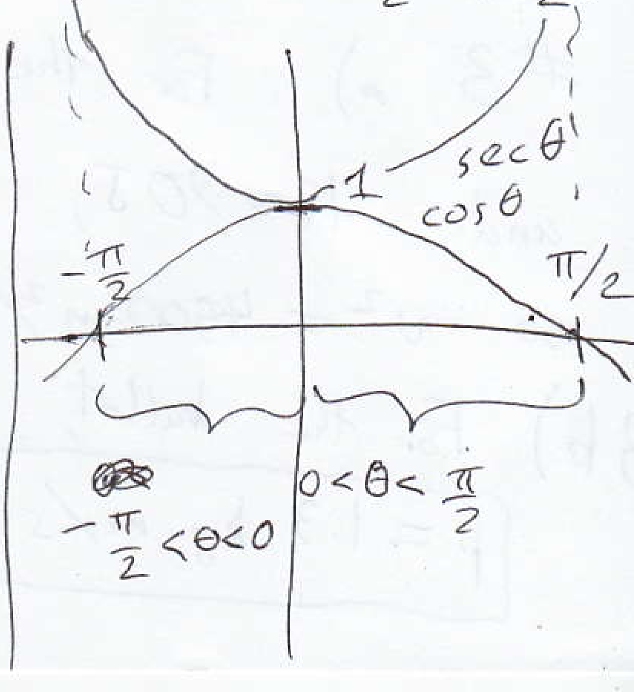
integrals like $\int\sqrt{1-x^2}\,dx$

Now, use $\sqrt{\sec^2\theta - 1} = \tan\theta$ for

$\boxed{\text{for } 0<\theta<\frac{\pi}{2}}$ integrals like $\int\sqrt{x^2-1}\,dx$

Always choose θ between $-\frac{\pi}{2}$ & $\frac{\pi}{2}$.





$$I = \int_{50}^{100} \frac{dx}{\sqrt{3x^2+4x-50}} = ?$$

$$I = \frac{1}{\sqrt{3}} \int_{50}^{100} \frac{dx}{\sqrt{\underbrace{x^2+(4/3)x - 50/3}_{\text{complete the square}}}}$$

$$x^2+px = \left(x+\frac{p}{2}\right)^2 - \left(\frac{p}{2}\right)^2$$

$$x^2+\frac{4}{3}x = \left(x+\frac{2}{3}\right)^2 - \left(\frac{2}{3}\right)^2$$

$$x^2+\frac{4}{3}x-\frac{50}{3} = \left(x+\frac{2}{3}\right)^2 - \frac{4}{9} - \frac{150}{9}$$

$$I = \frac{1}{\sqrt{3}} \int_{50}^{100} \frac{dx}{\sqrt{(x+2/3)^2 - 154/9}}$$

$$I = \frac{1}{\sqrt{3}\sqrt{154/9}} \int_{50}^{100} \frac{dx}{\sqrt{\underbrace{\underbrace{(9/154)(x+2/3)^2}_{\sec^2\theta} - 1}_{\sqrt{\sec^2\theta - 1} = \tan\theta}}}$$

$$\frac{9}{154}\left(x+\frac{2}{3}\right)^2 = \sec^2\theta$$

Pick $\sqrt{\frac{9}{154}}\left(x+\frac{2}{3}\right) = \sec\theta$ (see below)

$0 \le \theta \le \pi$

and restrict θ.

Check that actually $0 < \theta < \frac{\pi}{2}$:

$x = 100 \Rightarrow \sqrt{\frac{9}{154}}\left(100+\frac{2}{3}\right) = \sec\theta = \frac{1}{\cos\theta}$

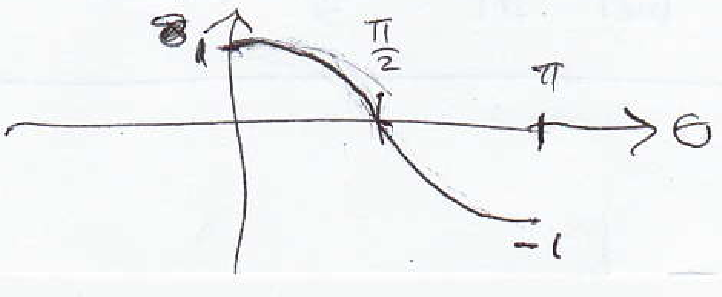
$x = 50 \Rightarrow \sqrt{\frac{9}{154}}\left(50+\frac{2}{3}\right) = \sec\theta = \frac{1}{\cos\theta}$

$x = 100 \Rightarrow \sqrt{\frac{154}{9}}\,\frac{1}{\left(100+\frac{2}{3}\right)} = \cos\theta$

$x = 50 \Rightarrow \underbrace{\sqrt{\frac{154}{9}}\,\frac{1}{(50+2/3)}} = \cos\theta$

$x = 50, 100 \Rightarrow$ both positive, so $0 < \theta < \frac{\pi}{2}$

If they were both negative, you would want to restrict θ to be between $\frac{\pi}{2}$ & π.



The arccos, or $\cos^{-1}$, function gives the unique θ-solution to

$y = \cos\theta$ for $-1 \le y \le 1$ and

$0 \le \theta \le \pi$

$$\frac{1}{2} = \cos\theta \ \& \ 0 \le \theta \le \pi$$

$$\Rightarrow \theta = \arccos\frac{1}{2} = 60° = \frac{\pi}{3}$$

$$x = 100 \Rightarrow \underbrace{\arccos\left(\sqrt{\frac{154}{9}}\,\frac{1}{(100+2/3)}\right)}_{\beta} = \theta$$

$$x = 50 \Rightarrow \underbrace{\arccos\left(\sqrt{\frac{154}{9}}\,\frac{1}{(50+2/3)}\right)}_{\alpha} = \theta$$

$$\sqrt{\frac{9}{154}}\left(dx + 0\right) = \sec\theta\tan\theta\, d\theta$$

$$dx = \sqrt{\frac{154}{9}}\sec\theta\tan\theta\, d\theta$$

(see next page for more steps)

$$\sqrt{\frac{9}{154}}\left(x+\frac{2}{3}\right)=\sec\theta$$

$$d\left(\sqrt{\frac{9}{154}}\left(x+\frac{2}{3}\right)\right)=d(\sec\theta)$$

$$\sqrt{\frac{9}{154}}\left(dx+0\right)=\underbrace{\boxed{\frac{d(\sec\theta)}{d\theta}}}\,d\theta$$

$$\sec\theta\tan\theta=(\sec\theta)'$$

$$I = \frac{1}{\sqrt{3}\cancel{\sqrt{154/9}}} \int_\alpha^\beta \frac{\cancel{\sqrt{154/9}} \sec\theta \cancel{\tan\theta}\, d\theta}{\cancel{\tan\theta}}$$

$$I = \frac{1}{\sqrt{3}} \int_\alpha^\beta \sec\theta \, d\theta = \frac{1}{\sqrt{3}} \ln|\sec\theta + \tan\theta| \Big|_\alpha^\beta$$

$$I = \frac{1}{\sqrt{3}} \left(\ln|\sec\beta + \tan\beta| - \ln|\sec\alpha + \tan\alpha| \right)$$

$$I = \boxed{\frac{1}{\sqrt{3}} \ln \left| \frac{\sec\beta + \tan\beta}{\sec\alpha + \tan\alpha} \right|}$$

$$\int \frac{dx}{x\sqrt{x^2-1}} = \int \frac{\cancel{\sec\theta} \cancel{\tan\theta}\, d\theta}{\cancel{\sec\theta} \cancel{\tan\theta}} = \theta + c$$

$$\sqrt{\sec^2\theta - 1} = \tan\theta$$

Pick $x = \sec\theta \Rightarrow \frac{1}{x} = \cos\theta \Rightarrow \arccos\left(\frac{1}{x}\right) = \theta$

$dx = \sec\theta \tan\theta \, d\theta$

restrict θ to be between 0 & π.

$$\int \frac{dx}{x\sqrt{x^2-1}} = \arccos\left(\frac{1}{x}\right) + c$$

$$\frac{1}{x\sqrt{x^2-1}} = \left(\arccos\left(\frac{1}{x}\right)\right)'$$

HW #1 $\displaystyle\int \frac{dx}{x^2\sqrt{x^2-1}} = ?$

#2 $\displaystyle\int_2^3 \sqrt{x^2-1}\, dx = ?$

#3 $\displaystyle\int_5^8 \sqrt{2x^2+x-8}\, dx = ?$

$\cos\theta = \sqrt{1-\sin^2\theta}$ for $-\frac{\pi}{2} \le \theta \le \frac{\pi}{2}$

Use sine substitution for $\sqrt{1-x^2}$, $\frac{1}{1-x^2}$, etc...

$$\begin{cases} \tan\theta = \sqrt{\sec^2\theta - 1} & \text{for } 0 \le \theta < \frac{\pi}{2} \\ \tan\theta = -\sqrt{\sec^2\theta - 1} & \text{for } \frac{\pi}{2} < \theta \le \pi \end{cases}$$

Use secant substitution for $\sqrt{x^2-1}$, $\frac{1}{x^2-1}$, etc...

$\sec\theta = \sqrt{1+\tan^2\theta}$ for $-\frac{\pi}{2} < \theta < \frac{\pi}{2}$

Use tangent substitution for $\sqrt{1+x^2}$, $\frac{1}{1+x^2}$, etc...