For Test #4:

Compute exact formulas for integrals using trigonometric techniques

- $\int \sin^m x \cos^n x \, dx = ?$
- $\int \tan^m x \sec^n x \, dx = ?$
- Trigonometric substitution for $\sqrt{x^2-1}$, $\sqrt{1-x^2}$, $\sqrt{x^2+1}$, etc
  $x = \sec\theta$ $x = \sin\theta$ $x = \tan\theta$
- Apply these technique to geometry problems.
- Reference materials (I'll email these)
  1. Table of integrals in back of book.
  2. Table of formulas for solids of revolutions.
- Open ~~to~~ notes, not calculators.

$$\pm\sqrt{\sec^2\theta - 1} = \tan\theta \qquad 0 \le \theta \le \pi$$

$$\sqrt{1-\sin^2\theta} = \cos\theta \qquad -\frac{\pi}{2} \le \theta \le \frac{\pi}{2}$$

$$\sqrt{\tan^2\theta + 1} = \sec\theta \qquad -\frac{\pi}{2} < \theta < \frac{\pi}{2}$$

$$\int \frac{dx}{x^2+1} = \int \frac{\cancel{\sec^2\theta}\, d\theta}{\cancel{\sec^2\theta}} = \theta + c$$

$$= \arctan x + c$$

$$\tan^2\theta + 1 = \sec^2\theta$$

$$x = \tan\theta$$

$$dx = \sec^2\theta \, d\theta$$

$$\int \frac{dx}{(x^2+1)^2} = \int \cos^2\theta \, d\theta = \ldots$$

$x = \tan\theta$

$$I = \int \sin^3 2x \cos^2 2x \, dx = ?$$

$$= \int \sin^{3} u \cos^2 u \, (du/2)$$

odd

$$u = 2x \Rightarrow du = 2dx \Rightarrow dx = du/2$$

$$I = \int \underbrace{\sin^2 u}_{1-w^2} \underbrace{\cos^2 u}_{w^2} \underbrace{\sin u \, du}_{-dw} /2$$

$w = \cos u$ $\qquad dw = -\sin u \, du$

$$-dw = \sin u \, du$$

$$\sin^2 u + \cos^2 u = 1$$

$$\sin^2 u + w^2 = 1$$

$$\sin^2 u = 1 - w^2$$

$$I = \int (1-w^2) w^2 (-dw)/2$$

$$I = \frac{1}{2}\int (-w^2 + w^4) \, dw = \frac{1}{2}\left(-\frac{w^3}{3} + \frac{w^5}{5}\right) + c$$

$$I = \frac{1}{2}\left(-\frac{\cos^3 u}{3} + \frac{\cos^5 u}{5}\right) + c$$

$$I = \boxed{\frac{1}{2}\left(-\frac{\cos^3 2x}{3} + \frac{\cos^5 2x}{5}\right) + c}$$

$$I = \int \tan^5 x \sec^4 x \, dx$$

Either use $\begin{cases} du = \sec^2 x \, dx \\ \text{or } du = \sec x \tan x \, dx \end{cases}$

Pick $du = \sec^2 x \, dx$ $\quad u = \tan x$

$$I = \int \underbrace{\tan^5 x}_{u^5} \underbrace{\sec^2 x}_{1+u^2} \underbrace{\sec^2 x \, dx}_{du} = \int u^5(1+u^2)\,du$$

$$\frac{\cos^2 x}{\cos^2 x} + \frac{\sin^2 x}{\cos^2 x} = \frac{1}{\cos^2 x}$$

$$I = \int (u^5 + u^7)\,du$$

$$I = \frac{u^6}{6} + \frac{u^8}{8} + c$$

$$1 + u^2 = 1 + \tan^2 x = \sec^2 x$$

$$I = \boxed{\frac{\tan^6 x}{6} + \frac{\tan^8 x}{8} + c}$$

$\int \tan^{\text{even}} x \sec^{\text{odd}} x \, dx$ is harder.

Remember $\int \sec x \, dx$ & $\int \sec^3 x \, dx$
(or put in notes)

$\int \cos^{even} x \ \sin^{even} x \ dx$

Use half-angle formulas.

$$\int \cos^2 x \sin^2 x \, dx = \int \left(\frac{1+\cos 2x}{2}\right)\left(\frac{1-\cos 2x}{2}\right) dx$$

$$= \int \frac{1^2 - \cos^2 2x}{4} \, dx \qquad (a+b)(a-b) = a^2 - b^2$$

$$= \int \left(\frac{1}{4} - \frac{1}{4}\cos^2 2x\right) dx$$

$$= \int \left(\frac{1}{4} - \frac{1}{4}\left(\frac{1+\cos 4x}{2}\right)\right) dx$$

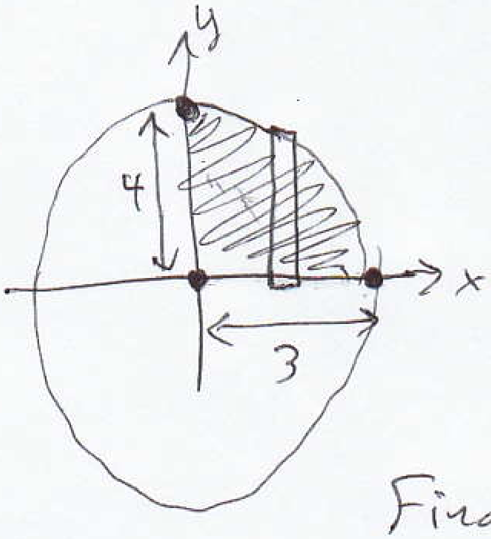
$$= \int \left(\frac{1}{4} - \frac{1}{8} - \frac{1}{8}\cos 4x\right) dx$$

$u = 4x \quad du = 4dx \quad dx = du/4$

$$= \int \left(\frac{1}{8} - \frac{1}{8}\cos u\right)\left(\frac{du}{4}\right)$$

$$= \left(\frac{u}{8} - \frac{1}{8}\sin u\right)\left(\frac{1}{4}\right) + c$$

$$= \boxed{\left(\frac{4x}{8} - \frac{1}{8}\sin 4x\right)\left(\frac{1}{4}\right) + c}$$



Ellipse

$$\frac{x^2}{3^2} + \frac{y^2}{4^2} = 1 \qquad y > 0$$

Find exact area of a quadrant.

$$A = \int_{x=0}^{x=3} y \, dx \qquad \frac{y^2}{4^2} = 1 - \frac{x^2}{3^2}$$

$$A = \int_0^3 4\underbrace{\sqrt{1 - x^2/9}}\, dx \qquad y^2 = 16\left(1 - x^2/9\right)$$

$$\cos^2\theta = 1 - \sin^2\theta \qquad y = 4\sqrt{1 - x^2/9}$$

$$-\frac{\pi}{2} \le \theta \le \frac{\pi}{2} \Rightarrow \cos\theta = \sqrt{1 - \sin^2\theta}$$

$$\sin^2\theta = \frac{x^2}{9}$$

$$\theta = \arcsin(x/3) \qquad \text{Pick } \boxed{\sin\theta = x/3}$$

$$\cos\theta \, d\theta = dx/3$$

$$x = 0 \Rightarrow \theta = \arcsin 0 = 0$$

$$3\cos\theta \, d\theta = dx$$

$$x = 3 \Rightarrow \theta = \arcsin 1 = \frac{\pi}{2}$$

$$(\sin\theta)' d\theta = (x/3)' dx$$

$$\sin 0 = 0$$

$$\sin \pi/2 = 1$$

$$\cos^2\theta$$

$$A = \int_0^{\pi/2} 4\cos\theta \cdot 3\cos\theta \, d\theta$$

$$A = 12\int_0^{\pi/2} \frac{1 + \cos 2\theta}{2} d\theta = 6\int_0^{\pi/2} (1 + \cos 2\theta) \, d\theta$$

$$A = 6\int_0^{\pi} (1+\cos u)(du/2) = 3(u+\sin u)\Big|_0^{\pi}$$

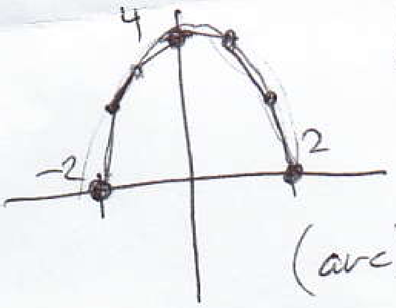
$u = 2\theta \qquad du = 2\,d\theta \qquad \cancel{du = 2\,d\theta}$

$du/2 = d\theta$

$\theta = 0 \Rightarrow u = 2\theta = 0$

$\theta = \pi/2 \Rightarrow u = 2\theta = \pi$

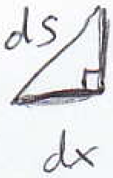
$$A = 3(\pi + \underbrace{\sin\pi}_{0} - 0 - \sin 0) = \boxed{3\pi}$$



$y = 4 - x^2$

$-2 \le x \le 2$

(arc) Length $L = ?$



$dy = (4-x^2)'\,dx = (0-2x)\,dx$

$ds = \sqrt{dx^2 + dy^2}$

$ds = \sqrt{dx^2 + (-2x)^2 dx^2}$

$ds = \sqrt{1+4x^2}\,\underbrace{\sqrt{dx^2}}_{= dx}$

because $dx > 0$ if we go the right

$L = 2\int_0^2 ds$

$L = 2\int_0^2 \sqrt{1+\underline{4x^2}}\,dx$

$1 + \underline{\tan^2\theta} = \sec^2\theta \qquad \sqrt{1+\tan^2\theta} = \sec\theta$

$-\frac{\pi}{2} < \theta < \frac{\pi}{2}$

$4x^2 = \tan^2\theta$

Pick $\boxed{2x = \tan\theta} \Rightarrow 2\,dx = \sec^2\theta\,d\theta$

$$x = 0 \Rightarrow 2 \cdot 0 = \tan\theta \Rightarrow 0 = \tan\theta \Rightarrow \theta = 0$$

$$x = 2 \Rightarrow 2 \cdot 2 = \tan\theta \Rightarrow \theta = \arctan(4)$$

$$L = 2\int_0^{\arctan(4)} \overbrace{\underbrace{\sec\theta}_{\sqrt{1+4x^2}} \; \underbrace{\sec^2\theta \, d\theta / 2}_{dx}}^{\sec^3\theta}$$

$$L = \cancel{2}\left(\frac{1}{2}\sec\theta\tan\theta + \frac{1}{2}\ln\left|\sec\theta + \tan\theta\right|\right)\Bigg|_0^{\arctan(4)}$$

$$\sec 0 = \frac{1}{\cos 0} = \frac{1}{1} = 1$$

$$\tan 0 = 0 \qquad \tan(\arctan(4)) = 4$$

$$\sec(\arctan(4)) = \sqrt{1+\tan^2(\arctan(4))}$$
$$= \sqrt{1+4^2} = \sqrt{17}$$

$$L = \frac{1}{2}\left[\sqrt{17}\cdot 4 + \ln\left|\sqrt{17}+4\right|\right] - \frac{1}{2}\left[\left(1\cdot 0 + \underbrace{\ln|1+0|}_{\ln 1 = 0}\right)\right]$$

$$L = 2\sqrt{17} + \frac{1}{2}\ln\left(4+\sqrt{17}\right)$$