time

t=0 @ 1988

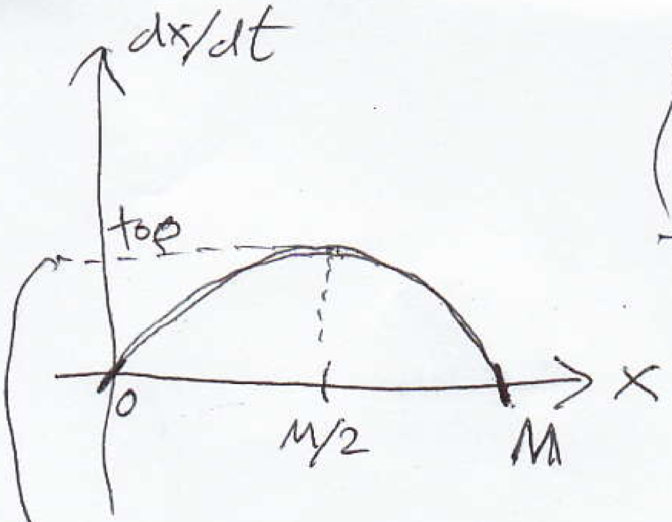
t	x (# cell phones / # people)	$\Delta x/\Delta t$	midpoint t	midpoint x
0	0.0000	0.0019	0.5	0.0095
1	0.0019	0.0021	1.5	0.0030
2	0.0040	0.0056	2.5	0.0068
3	0.0096	0.0089	3.5	0.0140
4	0.0185	0.0072	4.5	0.0221
5	0.0257	0.0019	5.5	0.0266
6	0.0276	0.0086	6.5	0.0319
7	0.0362	0.0089	7.5	0.0407
8	0.0451	0.0235	8.5	0.0568
9	0.0686	0.1470	9.5	0.1421
10	0.2156	0.3068	10.5	0.3690
11	0.5224	0.2800	11.5	0.6624
12	0.8024	0.1700	12.5	0.8874
13	0.9724	0.1106	13.5	1.0277
14	1.0830	0.0584	14.5	1.1122
15	1.1414	-0.1383	15.5	1.0722
16	1.0031	-0.0294	16.5	0.9884
17	0.9737	0.0423	17.5	0.9948
18	1.0160			

($\Delta t = 1$ always)

t=18 @ 2006

Actual data for Taiwan.

Plot: $\Delta x/\Delta t$



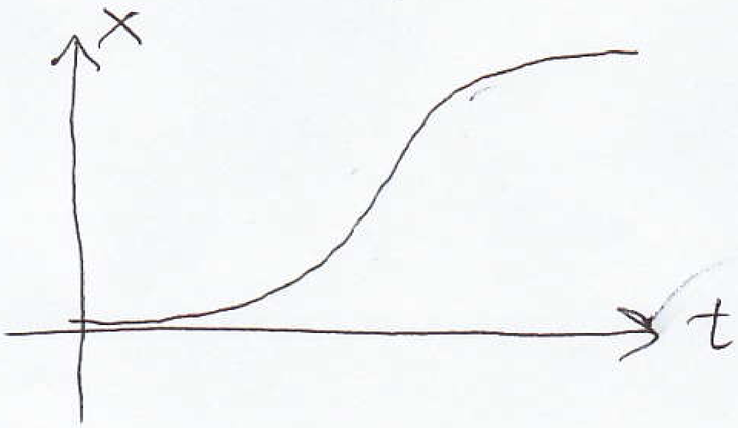
$$\boxed{\frac{dx}{dt} = kx(M-x)}$$

Differential equation

k some constant

$$\text{top} = k\left(\frac{M}{2}\right)\left(M-\frac{M}{2}\right) = \frac{kM^2}{4}$$

$$\frac{4 \cdot \text{top}}{M^2} = k$$



solution

$x = f(t)$

Also need to plug in one data point to determine a constant of integration.

$$\frac{dx}{dt} = kx(M-x)$$

$$dx = kx(M-x)\,dt$$

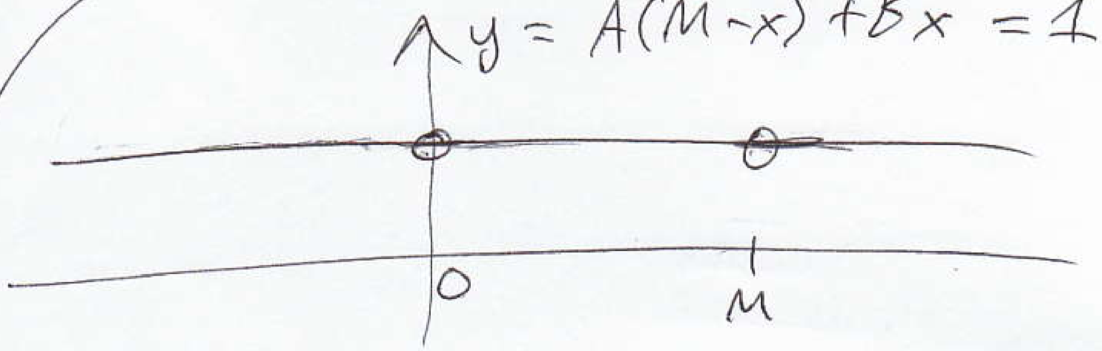
$$\int \frac{1\,dx}{x(M-x)} = \int k\,dt \quad \leftarrow \text{separable}$$

$$kt + c$$

$$\frac{A}{x} + \frac{B}{M-x} = \frac{1}{x(M-x)}$$

$$\frac{A(M-x)}{x(M-x)} + \frac{Bx}{x(M-x)} = \frac{1}{x(M-x)}$$

$$A(M-x) + Bx = 1$$



→ true at all x, even $0, M$.

$x=0 \Rightarrow A(M-0) + B\cdot 0 = 1 \Rightarrow AM=1$

$x=M \Rightarrow A(M-M) + B\cdot M = 1 \Rightarrow BM=1$

$A = 1/M \qquad B = 1/M$

$$\frac{1}{M}\ln|x| - \frac{1}{M}\ln|M-x| = \int \frac{dx}{x(M-x)}$$

$$\frac{1}{M}\left(\ln|x| - \ln|M-x|\right) = kt + c$$

$$\frac{1}{M}\ln\left|\frac{x}{M-x}\right| = kt + c$$

Partial Fractions: find constants A, B:

$$\frac{A}{x} + \frac{B}{M-x} = \frac{1}{x(M-x)}$$

In general, you can break up

$$\frac{ax+b}{(fx+g)(hx+k)} = \frac{A}{fx+g} + \frac{B}{hx+k}.$$

$$\int \frac{A}{x}\,dx + \int \frac{B\,dx}{M-x} = \int \frac{1\,dx}{x(M-x)}$$

$$A \ln|x| + \int \frac{B(-du)}{u} = \int \frac{dx}{x(M-x)}$$

$u = M-x$
$du = -dx$
$-du = dx$

$$\to A \ln|x| - B \ln|u|$$

$$\to A \ln|x| - B \ln|M-x|$$

Let's find A & B...

$$\ln\left|\frac{x}{M-x}\right| = M(kt+c)$$

$$\left|\frac{x}{M-x}\right| = e^{M(kt+c)}$$

If $0 < x < M$, then $\begin{cases} x > 0 \\ M - x > 0 \end{cases}$,

so $\frac{x}{M-x} > 0$, so $\left|\frac{x}{M-x}\right| = \frac{x}{M-x}$.

$$\frac{x}{M-x} = e^{M(kt+c)}$$

$$x = (M-x)\, e^{M(kt+c)}$$

$$1\cdot x = Me^{M(kt+c)} - xe^{M(kt+c)}$$

$$x\left(1+e^{M(kt+c)}\right) = Me^{M(kt+c)}$$

$$x = \frac{Me^{M(kt+c)}}{1+e^{M(kt+c)}}$$

$$x = \frac{M}{e^{-M(kt+c)}+1}$$

HW:

Solve $$\frac{dx}{dt} = \frac{3}{5x(x+2)}$$

assume $x > 0$

FYI: Look up "logistic curve"
"technology diffusion"
for more examples
of applications.