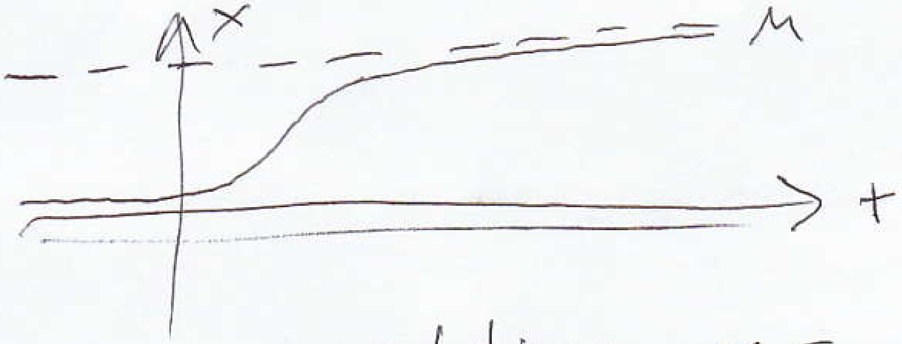
Today: more on differential equations (14.1 in book)

Monday: logistic growth $\begin{cases} 0 < x < M \\ \dfrac{dx}{dt} = kx(M-x) \end{cases}$

k, M positive constants



solutions $x = \dfrac{M}{1+e^{-M(kt+c)}}$

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exponential growth $\dfrac{dx}{dt} = kx$

k positive constant

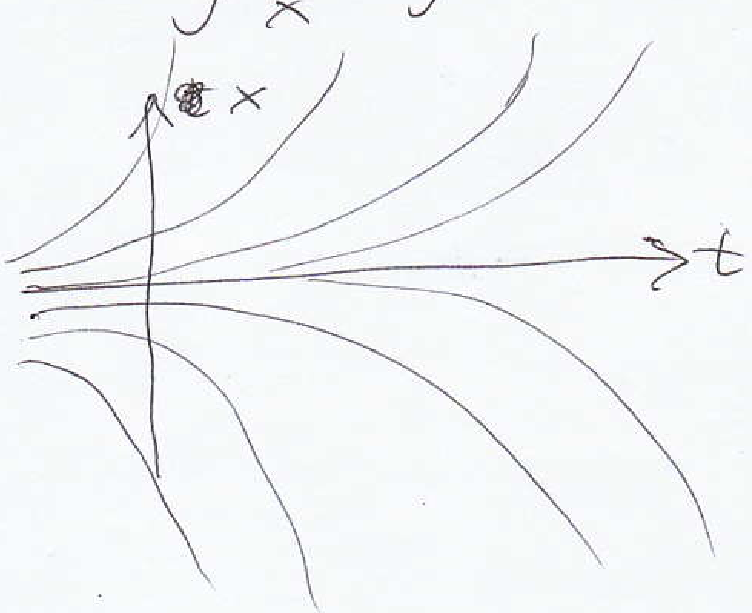
$dx = kx\,dt$

$\int \frac{dx}{x} = \int k\,dt$ $\qquad \ln|x| = kt + c$

$|x| = e^{kt+c}$

$x = \pm e^{kt+c}$

$x = \underbrace{(\pm e^{c})}_{A} e^{kt}$



$\boxed{x = Ae^{kt}}$

exponential decay $\quad \dfrac{dx}{dt} = \underbrace{-k}x$

solutions:

$$x = Ae^{-kt}$$

$-k$ is a negative constant

Suppose a population of bacteria is 15,000 at noon today and 17,000 at 1PM today.

Assuming exponential growth, what is the population at 3PM?

~~Assuming~~ Suppose that instead of the exp. growth prediction, it's actually 18,000 at 3PM. Use a logistic growth to predict the population at 4PM.

$$\frac{dx}{dt} = kx(M-x) \qquad x = \frac{M}{1+e^{-M(kt+c)}}$$

$M, k, c = ?$

t	x
0	15000
1	17000
3	18000

hard to solve for M, k, c...

$$\int \frac{dx}{x(M-x)} = \int k\,dt$$

$$\frac{1}{M}\int \frac{dx}{x} + \frac{1}{M}\int \frac{dx}{M-x} = kt + c$$

$$\frac{1}{M}\ln|x| + \frac{1}{M}\ln|M-x| = kt + c$$

$$\frac{1}{M}\ln\left(x(M-x)\right) = kt + c$$

$$\frac{1}{M}\ln\left(17000(M-17000)\right) = k \cdot 1 + c$$

$$\frac{1}{M}\ln\left(15000(M-15000)\right) = k \cdot 0 + c$$

subtract bottom from top

$$\frac{1}{M}\ln\frac{17000(M-17000)}{15000(M-15000)} = k$$

$$\frac{1}{M}\ln\left(15000(M-15000)\right) = c$$

$t=0$ is noon $\qquad t=1$ is 1PM

$$\frac{dx}{dt} = kx \qquad x = Ae^{kt}$$

$$t=0 \Rightarrow 15000 = x = Ae^{k\cdot 0} = Ae^{0} = A$$

$$t=1 \Rightarrow 17000 = x = Ae^{k\cdot 1} = Ae^{k} = 15000e^{k}$$

$$\frac{17}{15} = e^{k} \qquad k = \ln\frac{17}{15}$$

$t=3$ is 3PM: $\quad x = Ae^{k\cdot 3}$

$$x = 15000(e^{k})^{3} = 15000\left(\frac{17}{15}\right)^{3} = 21,835$$

$$\frac{1}{M}\ln(18000(M-18000)) = k\cdot 3 + c$$

subtract: $$\frac{1}{M}\ln(15000(M-15000)) = c$$

$$\frac{1}{M}\ln\frac{18000(M-18000)}{15000(M-15000)} = 3k =$$

$$\frac{3}{M}\ln\frac{17000(M-17000)}{15000(M-15000)}$$

$$\ln\frac{18000(M-18000)}{15000(M-15000)} = 3\ln\frac{17000(M-17000)}{15000(M-15000)}$$

$$\cancel{\ln}\frac{18000(M-18000)}{15000(M-15000)} = \cancel{\ln}\left(\left(\frac{17000(M-17000)}{15000(M-15000)}\right)^3\right)$$

$$\frac{18(M-18000)}{15(M-15000)} = \left(\frac{17(M-17000)}{15(M-15000)}\right)^3$$

$$[18(M-18000)][15^3(M-15000)^3]$$

$$= [15(M-15000)][17^3(M-17000)^3]$$

$$M \approx 18,200$$

At 5PM, logistic growth predicts:

$$x = \frac{M}{1+e^{-M(kt+c)}}$$

$$k = \frac{1}{18200} \ln \frac{17 \cdot 1200}{15 \cdot 3200}$$

$$c = \frac{1}{18200} \ln (15000 \cdot 3200)$$

$M = 18,200$

$t = 5$

$k \approx 4.7 \times 10^{-5}$

$c \approx 9.7 \times 10^{-4}$

sign errors?

~~Possible M was $\approx 30,000$?~~

$x = 18,199$ at 5PM

Consider $\dfrac{dx}{dt} = kx^2$ & $x > 0$

k positive constant

"hyperbolic growth"

$x > 0$ & $\dfrac{dx}{dt} = -kx^2$ hyperbolic decay

HW#1 If $\frac{dx}{dt} = kx^2$, $x = 100$ at $t=0$,

and $x = 200$ at $t=3$,

when does $x = 400$?

When does $x \to \infty$?

HW#2 If $\frac{dx}{dt} = kx(M-x)$

and $M = 1500$,

$x = 20$ at $t=1$,

$x = 30$ at $t=2$,

$x = ?$ at $t = 30$.

HW#3 If $\frac{dx}{dt} = \frac{x^2+1}{3e^t}$

and $x = 1$ when $t=0$,

then find a formula for x

as a function of t.