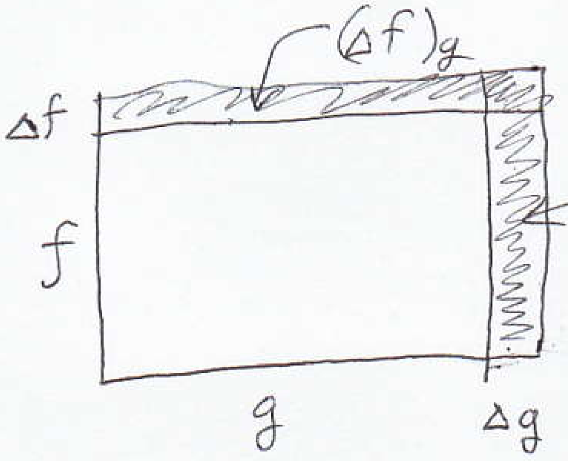
Today: Integration by Parts

Product rule $(fg)' = f'g + fg'$



$$d(fg) = (df)g + f\,dg$$

$$\Delta(fg) \approx (\Delta f)g + f(\Delta g)$$

$$d(fg) = g\,df + f\,dg$$

$$d(uv) = v\,du + u\,dv$$

$$uv + c = \int d(uv) = \int v\,du + \int u\,dv$$

put the $+c$ inside $\int v\,du$

$$uv = \int v\,du + \int u\,dv$$

$$uv - \int v\,du = \int u\,dv$$

$$\int u\,dv = uv - \int v\,du$$

Integration by parts

$$\int \underbrace{x}_{u}\,\underbrace{e^x dx}_{dv} = \underbrace{x}_{u}\underbrace{e^x}_{v} - \int \underbrace{e^x}_{v}\,\underbrace{dx}_{du} = \boxed{xe^x - e^x + c}$$

I picked $u = x \quad dv = e^x dx$

$v = \int dv = \int e^x dx = e^x$ (leaving out $+c$)

IBP helps when you have, for example, $\int x^m e^{kx} dx$, $\int x^m \sin(kx) dx$, etc...

because we can integrate e^{kx} or $\sin(kx)$, etc as many times as we need to, and differentiate x^m, until it's ~~a~~ a constant.

$$\int \underbrace{x^2}_{u} \underbrace{\sin x \, dx}_{dv} = uv - \int v \underbrace{du}_{2x\,dx}$$

$$v = -\cos x$$

$$\int x^2 \sin x \, dx = x^2(-\cos x) - \int (-\cos x)(2x \, dx)$$
$$= -x^2 \cos x + 2\int x \cos x \, dx$$

$$\int \underbrace{x}_{u} \underbrace{\cos x \, dx}_{dv} = uv - \int v \underbrace{du}_{dx} = x \sin x - \underbrace{\int \sin x \, dx}_{-\cos x}$$

new u, v $\qquad v = \sin x$

$$\int \sin x \, dx = \int d(-\cos x) = -\cos x + c$$

$$\int x \cos x \, dx = x \sin x + \cos x + c$$

$$\int x^2 \sin x \, dx = -x^2 \cos x + 2(x \sin x + \cos x) + c \quad \checkmark$$

$$\begin{array}{c|cc|c}
 & x^2 & \sin x & \\
d \downarrow & 2x & -\cos x & \int \downarrow \\
 & 2 & -\sin x & \\
 & 0 & \cos x &
\end{array}$$

Alternating $+, -, +, -, \ldots$

$+(-x^2 \cos x)$

$-(2x(-\sin x))$

$+(2 \cos x)$

stop at 0

$$-x^2 \cos x + 2x \sin x + 2 \cos x + c \quad \checkmark$$

$$\int e^{3x}\underbrace{x^4}_{u}\,dx = \int \underbrace{x^4}_{u}\,\underbrace{e^{3x}\,dx}_{dv}$$

$$\begin{array}{c|cc|cc}
 & x^4 & e^{3x} & & \\
d & 4x^3 & e^{3x}/3 & \int & + \\
 & 12x^2 & e^{3x}/9 & & - \\
 & 24x & e^{3x}/27 & & + \\
 & 24 & e^{3x}/81 & & - \\
 & 0 & e^{3x}/243 & & +
\end{array}$$

$$\int e^{3x}\,dx = \int e^{w}\,\frac{dw}{3}$$

$w = 3x$

$dw = 3dx$

$dw/3 = dx$

$\frac{1}{3}e^{w} + c$

$\frac{1}{3}e^{3x} + c$

$$\int e^{3x}x^4\,dx = \frac{x^4}{3}e^{3x} - \frac{4x^3}{9}e^{3x} + \frac{12x^2}{27}e^{3x}$$

$$- \frac{24x}{81}e^{3x} + \frac{24}{243}e^{3x} + c$$

$$= e^{3x}\left(\frac{x^4}{3} - \frac{4x^3}{9} + \frac{12x^2}{27} - \frac{24x}{81} + \frac{24}{243}\right) + c$$

Let's check our answer

$$\left[e^{3x}\left(\frac{x^4}{3} - \frac{4x^3}{9} + \frac{12x^2}{27} - \frac{24x}{81} + \frac{24}{243}\right) + c\right]' =$$

$$=\left(e^{3x}\right)'\left(\frac{x^4}{3}-\frac{4x^3}{9}+\frac{12x^2}{27}-\frac{24x}{81}+\frac{24}{243}\right)$$

$$+\left(e^{3x}\right)\left(\frac{x^4}{3}-\frac{4x^3}{9}+\frac{12x^2}{27}-\frac{24x}{81}+\frac{24}{243}\right)'$$

$$=\left(3e^{3x}\right)\left(\frac{x^4}{3}-\frac{4x^3}{9}+\frac{12x^2}{27}-\frac{24x}{81}+\frac{24}{243}\right)$$

$$+\left(e^{3x}\right)\left(\frac{4x^3}{3}-\frac{12x^2}{9}+\frac{24x}{27}-\frac{24}{81}+0\right)$$

$$=e^{3x}\left(x^4-\cancel{\frac{4x^3}{3}}+\cancel{\frac{12x^2}{9}}-\cancel{\frac{24x}{27}}+\cancel{\frac{24}{81}}\right.$$
$$\left.+\cancel{\frac{4x^3}{3}}-\cancel{\frac{12x^2}{9}}+\cancel{\frac{24x}{27}}-\cancel{\frac{24}{81}}\right)$$

$$=x^4e^{3x} \checkmark$$

Another use of IBP:

If you can do $\int f(x)\,dx$

you can do $\int f^{-1}(x)\,dx$.

We know $\int \tan x \, dx = \ln|\sec x| + c$

Let's find $\int \arctan x \, dx$

Recall $\theta = \arctan x$ means $\begin{cases} x = \tan\theta \\ -\frac{\pi}{2} < \theta < \frac{\pi}{2} \end{cases}$

$x = \tan\theta \Rightarrow dx = \sec^2\theta \, d\theta$

$$\int \underbrace{\arctan x}_{\theta} \, dx = \int \underbrace{\theta}_{u} \underbrace{\sec^2\theta \, d\theta}_{dv}$$

$v = \tan\theta \qquad du = d\theta$

$$\int u \, dv = uv - \int v \, du = \theta \tan\theta - \int \tan\theta \, d\theta$$

$$= \theta \tan\theta - \ln|\sec\theta| + c$$

$$= \underbrace{(\arctan x)}_{\theta} \underbrace{x}_{\tan\theta} - \ln\sqrt{1+x^2} + c$$

$$|\sec\theta| = \sqrt{\sec^2\theta} = \sqrt{1+\tan^2\theta} = \sqrt{1+x^2}$$

$$\int \arctan x \, dx = x \arctan x - \ln\sqrt{1+x^2} + c$$

$$\int \ln x \, dx = \int \underbrace{w}_{\ln x} \underbrace{e^w \, dw}_{dx}$$

$x = e^w \qquad w = \ln x \qquad dx = e^w \, dw$

$$\int \underbrace{w}_{u} \underbrace{e^w\, dw}_{dv} = uv - \int v\, du = we^w - \int e^w\, dw$$

$$v = e^w \qquad du = dw$$

$$= we^w - e^w + c =$$

$$\underbrace{(\ln x)}_{w}\underbrace{x}_{e^w} - \underbrace{x}_{e^w} + c$$

$$\int \ln x\, dx = x \ln x - x + c$$

$$(x \ln x - x + c)'$$

$$x' \ln x + x (\ln x)' - 1$$

$$= \ln x + \underbrace{x\left(\frac{1}{x}\right)}_{1} - 1 = \ln x \checkmark$$

HW #1 $\int \arcsin x\, dx = ?$

#2 $\int x^3 (\sin 2x)\, dx = ?$

#3 $\int_0^1 x^2 e^{3x}\, dx = ?$

NEXT TEST APRIL 4 (Monday)