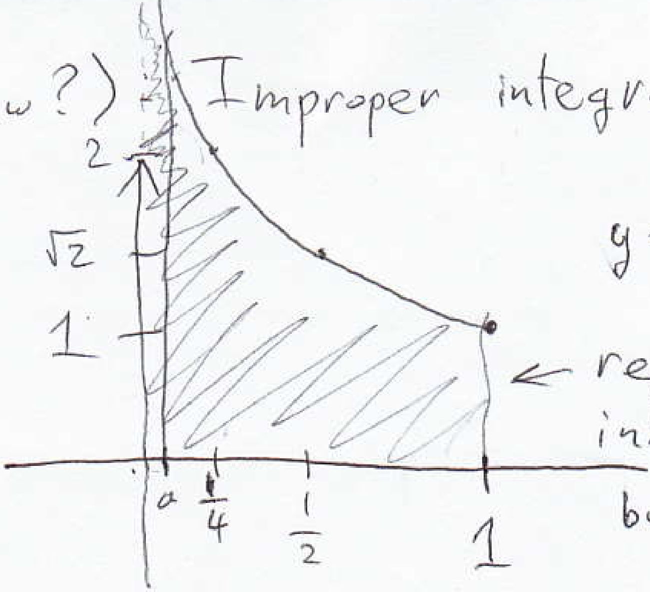
Today (& tomorrow?) Improper integrals

$$\int_0^1 \frac{dx}{\sqrt{x}} = ? \text{ ②}$$



$y = \frac{1}{\sqrt{x}}$

← region of infinite height, but area 2.

$\frac{1}{\sqrt{x}}$ is continuous

on $(0, 1]$,

but not on $[0, 1]$.

$$\frac{1}{\sqrt{1/n^2}} = \sqrt{n^2} = n$$

$$\frac{1}{\sqrt{+\text{small}}} = +\text{big}$$

$$\lim_{x \to 0^+} \frac{1}{\sqrt{x}} = \infty$$

We define $\int_0^1 \frac{dx}{\sqrt{x}} = \lim_{a \to 0^+} \int_a^1 \frac{dx}{\sqrt{x}}$

$$= \lim_{a \to 0^+} \int_a^1 x^{-1/2}\, dx = \lim_{a \to 0^+} \left. \frac{x^{-1/2+1}}{-1/2+1} \right|_a^1$$

$$= \lim_{a \to 0^+} \left. \frac{x^{1/2}}{1/2} \right|_a^1 = \lim_{a \to 0^+} \left. 2\sqrt{x} \right|_a^1$$

$$= \lim_{a \to 0^+} \left(2\sqrt{1} - 2\sqrt{a}\right) = \left(2\sqrt{1} - 2(0)\right) = 2$$

Limits Refence

±S for small; limit is 0

±M for medium; limit is not 0, not ±∞

±L for large; limit is ±∞

		b		
	a+b	+S	+M	+L
a	+S	+S	+M	+L
	+M	+M	+M	+L
	+L	+L	+L	+L

		b		
	a−b	+S	+M	+L
a	+S	±S	−M	−L
	+M	+M	±S ±M	−L
	+L	+L	+L	±S ±M ±L

		b		
	a·b	+S	+M	+L
	+S	+S	+S	+S +M +L
	+M	+S	+M	+L
	+L	+S +M +L	+L	+L

		b		
	a/b	+S	+M	+L
a	+S	+S +M +L	+S	+S
	+M	+L	+M	+S
	+L	+L	+L	+S +M +L

Example:

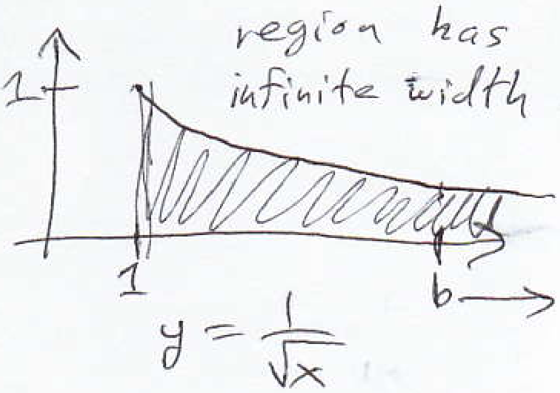
$$\lim_{x\to-\infty}\left(x^2-3x+2\right)=\lim_{x\to-\infty}x^2\left(1-\frac{3}{x}+\frac{2}{x^2}\right)$$

$$(-\text{large})(-\text{large})\left(1-\frac{3}{-\text{large}}+\frac{2}{\text{large}\cdot\text{large}}\right)$$

$$(+\text{large})\underbrace{\left(1+\text{small}+\text{small}\right)}_{\text{medium}}=+\text{large}$$

$\Rightarrow$ limit is ∞: $\lim_{x\to-\infty}\left(x^2-3x+2\right)=\infty$

$$A = \int_1^\infty \frac{dx}{\sqrt{x}} = \lim_{b \to \infty} \int_1^b \frac{dx}{\sqrt{x}}$$



$$A = \lim_{b \to \infty} \int_1^b x^{-1/2}\, dx$$

$\boxed{\text{infinite area}}$?

finite area?

$$= \lim_{b \to \infty} 2\sqrt{x} \Big|_1^b = \lim_{b \to \infty} (2\sqrt{b} - 2\sqrt{1}) = \infty$$

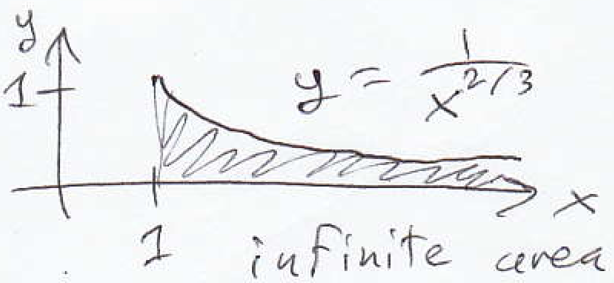
HW #1 Let $a > 0$.

For which values of p is $\int_a^\infty \frac{dx}{x^p}$ finite?

$$\int_1^\infty \frac{dx}{x^{2/3}} = \lim_{b \to \infty} \int_1^b x^{-2/3} dx = \lim_{b \to \infty} \frac{x^{-2/3+1}}{-2/3+1} \Big|_1^b$$

$$= \lim_{b \to \infty} \frac{x^{1/3}}{1/3} \Big|_1^b = \lim_{b \to \infty} 3\sqrt[3]{x} \Big|_1^b$$

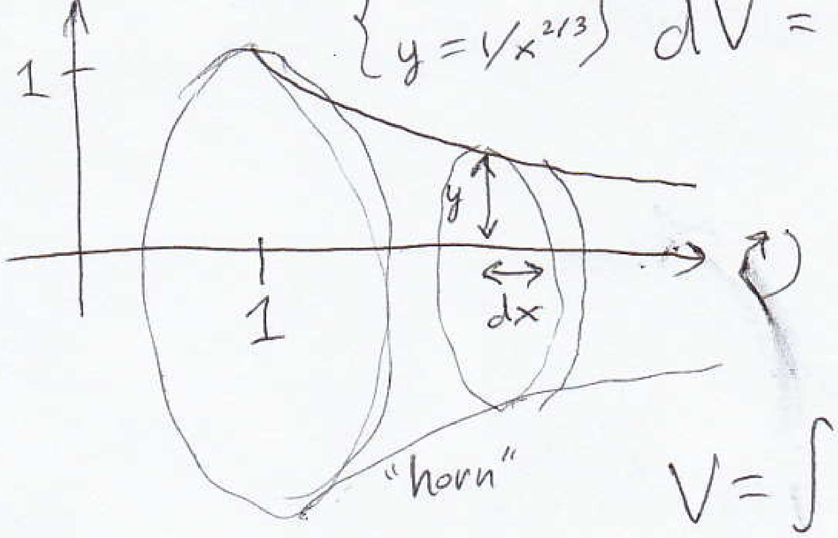
$$= \lim_{b \to \infty} \left(3\sqrt[3]{b} - 3\sqrt{1}\right) = \infty$$



Revolve around x-axis:

$y = \frac{1}{x^{2/3}}$; $x \geq 1$

$$\left\{\begin{matrix} x \ge 1 \\ y = 1/x^{2/3} \end{matrix}\right\} \quad dV = \pi y^2 \, dx$$



$$V = \int_{x=1}^{x=\infty} dV = \int_{x=1}^{x=\infty} \pi y^2 \, dx$$

$$V = \int_1^\infty \pi \left(\frac{1}{x^{2/3}}\right)^2 dx = \pi \int_1^\infty x^{-4/3} \, dx$$

$$= \text{[redacted]} \lim_{b \to \infty} \pi \int_1^b x^{-4/3} \, dx = \pi \lim_{b \to \infty} \left. \frac{x^{-4/3+1}}{-4/3+1} \right|_1^b$$

$$= \pi \lim_{b \to \infty} \left. \frac{x^{-1/3}}{-1/3} \right|_1^b = \pi \lim_{b \to \infty} \left(-3/\sqrt[3]{x}\right) \Big|_1^b$$

$$= \pi \lim_{b \to \infty} \left(\underbrace{\frac{-3}{\sqrt[3]{b}}}_{\downarrow} - \frac{-3}{\sqrt[3]{1}} \right) = \pi \left(\underset{\uparrow}{0} - \frac{-3}{\sqrt[3]{1}} \right)$$

$$\frac{-3}{\sqrt[3]{\text{large}}} = \frac{-3}{\text{large}} = -\text{small}$$

$$\boxed{V = 3\pi}$$

[redacted]

HW #2 Is the surface area of this "horn" finite or infinite?

Is $\int_5^\infty \sqrt{1+x^6}\,dx$ finite or infinite?

$$\sqrt{1+x^6} \geq \sqrt{x^6} = x^3 \geq 0 \quad (\text{when } x \geq 0),$$

$$\text{so } \int_5^\infty \sqrt{1+x^6}\,dx \geq \int_5^\infty x^3\,dx = \lim_{b\to\infty} \frac{x^4}{4}\Big|_5^b$$

More area under $y = \sqrt{1+x^6}$ than $y = x^3$

$$\int_5^\infty \sqrt{1+x^6}\,dx \geq \lim_{b\to\infty}\left(\frac{b^4}{4} - \frac{5^4}{4}\right) = \infty.$$

$$\text{So, } \int_5^\infty \sqrt{1+x^6}\,dx = \infty.$$

Limits Reference II

$\pm S$: small
$\pm M$: medium
$\pm L$: large

		b		
	a^b	+S	+M	+L
a	+S	+S +M	+S	+S
	+M; a>1	≈1	+M	+L
	+M; a=1	≈1	+M	1
	+M; a<1	≈1	+M	+S
	+L	+M +L	+L	+L

(For negative powers, use $a^{-b} = \frac{1}{a^b}$.)

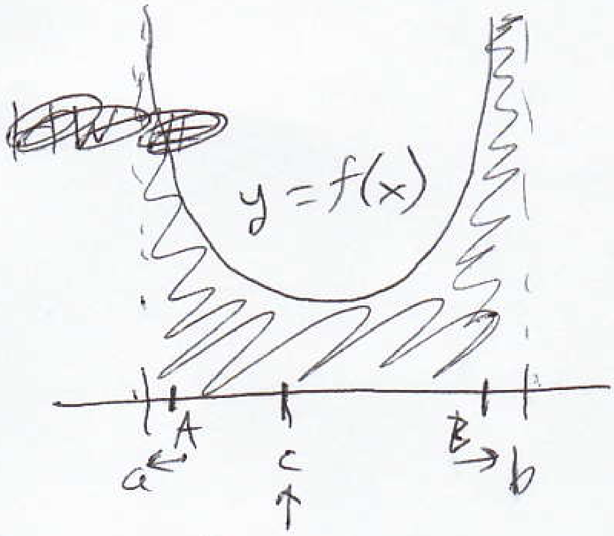
Why is $\sqrt{1+x^6} \geq (x-1)^3$ when $x \geq 5$?

$$\sqrt{1+x^6} \geq \sqrt{1-2x^3+x^6} = \sqrt{(x^3-1)^2} = x^3-1$$

$$\sqrt{1+x^6} \geq \sqrt{x^6} \geq \sqrt{(x-1)^6} = (x-1)^3$$

Actually, $\sqrt{1+x^6} \geq \sqrt{x^6} = x^3$ (when $x \geq 0$).

That would have been easier...



Pick any middle point

In this case, define

$$\int_a^b f(x)\,dx \quad \text{as}$$

$$\int_a^c f(x)\,dx + \int_c^b f(x)\,dx,$$

which is $\lim_{A \to a^+} \int_A^c f(x)\,dx + \lim_{B \to b^-} \int_c^B f(x)\,dx$

HW#3

Prove $\pi = \int_{-1}^{1} \frac{dx}{\sqrt{1-x^2}}$.