Today: Integral test (9.4)

Since last test:

Improper Integrals (6.7)
Geometric series (9.1, 9.2)
Ratio Test (9.6)
Power series (9.7, 9.8)

Tomorrow: review

Wednesday: Test 6

$$\int_0^x \frac{1}{1-t}\,dt = \int_0^x \sum_{k=0}^{\infty} t^k\,dt = \sum_{k=0}^{\infty} \int_0^x t^k\,dt$$

when $-1<x<1$

$$= \sum_{k=0}^{\infty} \left(\frac{t^{k+1}}{k+1} \Big|_0^x \right) = \sum_{k=0}^{\infty} \frac{x^{k+1}}{k+1}$$

What happens at $x=1$?

When $|x|>1$, the ratio test proves $\sum_{k=0}^{\infty} \frac{x^{k+1}}{k+1}$ diverges.

When $|x|<1$, the ratio test proves

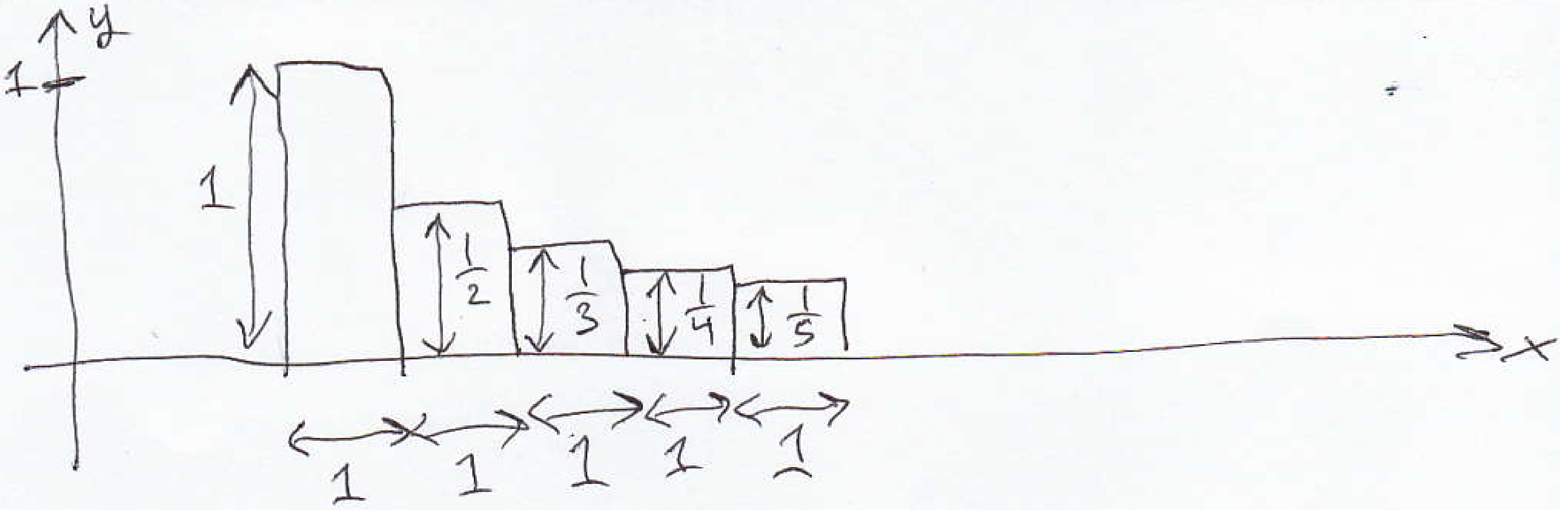
$$\sum_{k=0}^{\infty} \frac{x^{k+1}}{k+1} \quad \text{converges.}$$

This converges to $\int_0^x \frac{dt}{1-t}$

$$= -\ln(1-x).$$

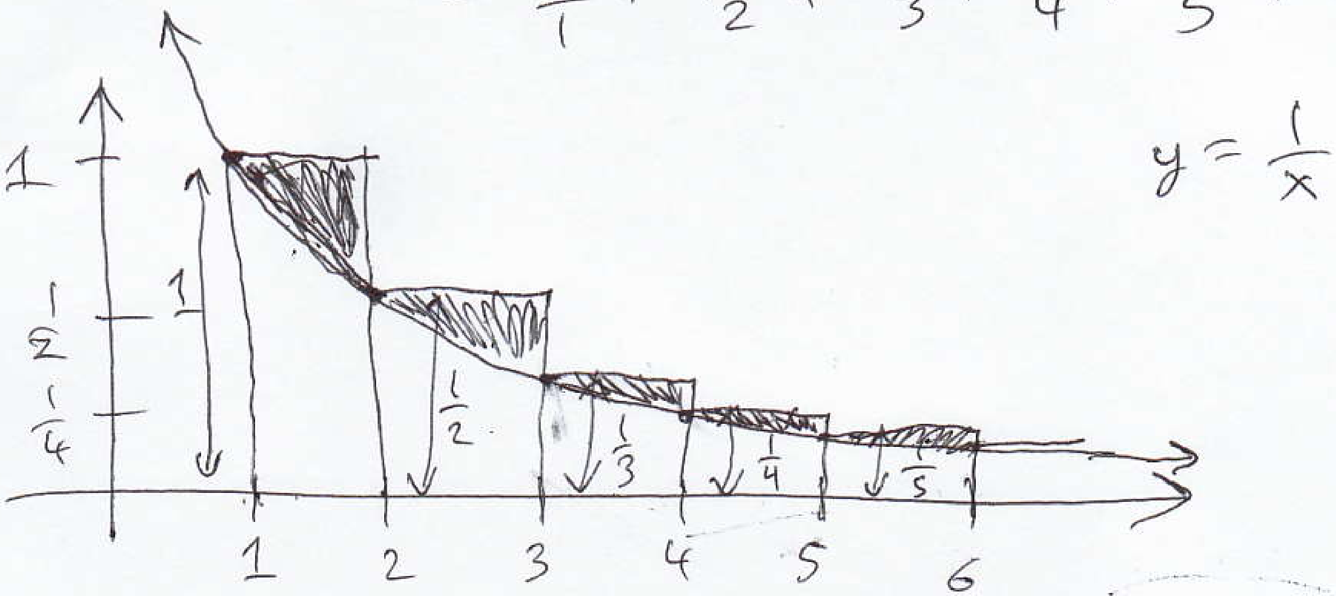
$$\sum_{k=0}^{\infty} \frac{1^{k+1}}{k+1} = \frac{1}{1} + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \frac{1}{5} + \frac{1}{6} + \frac{1}{7} + \cdots$$

(The ratio test is inconclusive...)



$$\text{area} = 1\cdot 1 + 1\cdot\frac{1}{2}\cdot 1\cdot\frac{1}{3} + 1\cdot\frac{1}{4} + 1\cdot\frac{1}{5} + \cdots$$

$$= \frac{1}{1} + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \frac{1}{5} + \cdots$$



$$1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \frac{1}{5} + \cdots \geq \int_1^\infty \frac{1}{x}\,dx$$

left-endpoint rule (with $\Delta x = 1$)
overestimate of integral

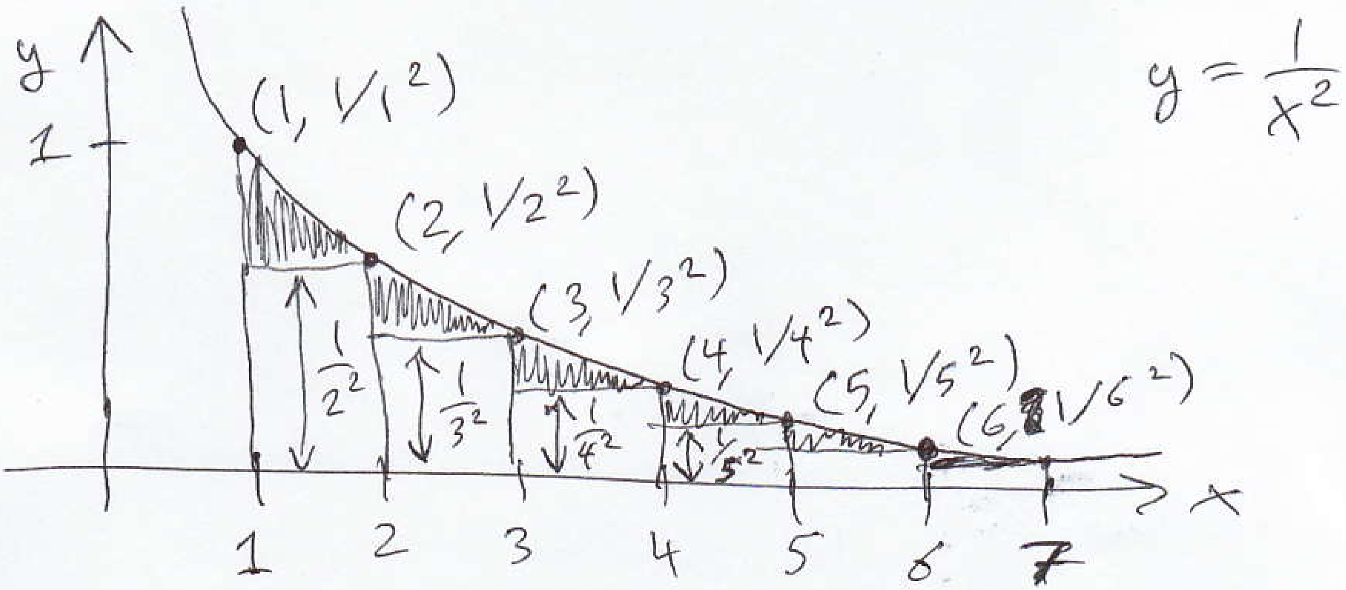
$$\int_1^\infty \frac{1}{x}\,dx = \lim_{z\to\infty} \int_1^z \frac{1}{x}\,dx = \lim_{z\to\infty} \Big(\ln|z| - \underbrace{\ln|1}_{0}$$

$$= \lim_{z\to\infty} \ln(z) = \infty$$

So, $1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \frac{1}{5} + \cdots = \infty.$

Does $\frac{1}{1^2} + \frac{1}{2^2} + \frac{1}{3^2} + \frac{1}{4^2} + \frac{1}{5^2} + \frac{1}{6^2} + \cdots$

converge?



$$\frac{1}{2^2} + \frac{1}{3^2} + \frac{1}{4^2} + \frac{1}{5^2} + \cdots \leq \int_1^\infty \frac{1}{x^2}\,dx$$

$$\frac{1}{1^2} + \frac{1}{2^2} + \frac{1}{3^2} + \frac{1}{4^2} + \frac{1}{5^2} + \cdots \leq \int_1^\infty \frac{dx}{x^2} + 1$$

$$\int_1^\infty \frac{dx}{x^2} = \lim_{z \to \infty} \int_1^z x^{-2}\,dx = \lim_{z \to \infty} \left.\frac{x^{-2+1}}{-2+1}\right|_1^z$$

$$= \lim_{z \to \infty} \frac{x^{-1}}{-1}\Big|_1^z = \lim_{z \to \infty} \frac{-1}{x}\Big|_1^z$$

$$= \lim_{z \to \infty} \left(\frac{-1}{z} - \frac{-1}{1} \right) = 0 - \frac{-1}{1} = 1$$

$$\int_1^\infty \frac{dx}{x^2} = 1$$

$$\underbrace{\frac{1}{1^2} + \frac{1}{2^2} + \frac{1}{3^2} + \frac{1}{4^2} + \frac{1}{5^2} + \cdots}_{\text{converges.}} \leq 1 + 1$$

HW: For which p does $\displaystyle\sum_{k=1}^{\infty} \frac{1}{k^p}$ converge? p can be any real.

Integral Test:

If f is continuous, nonnegative, and decreasing ($\searrow$) on $[m, \infty)$, then either $\sum_{k=m}^{\infty} f(k)$ & $\int_m^\infty f(x)\,dx$ both converge or both diverge.

This is true because:

$$\sum_{k=m+2}^{\infty} f(k) \le \int_{m+1}^{\infty} f(x)\,dx \le \sum_{k=m+1}^{\infty} f(k) \le \int_{m}^{\infty} f(x)\,dx$$