Improper integrals (6.7)

Series, esp. geometric series (9.1, 9.2)

Ratio test (9.6)

Power series (9.7, 9.8)

Integral Test (9.4)

Also, limits involving ∞ &
l'Hospital's rule are in 5.1, 5.2

$$\underbrace{\sum_{k=0}^{\infty} x^k}_{\text{Geometric series}} \begin{cases} \text{converges} & \text{if } -1 < x < 1 \\ \text{diverges} & \text{if } x \leq -1 \text{ or } x \geq 1 \end{cases}$$

$$\lim_{k \to \infty} \left| \frac{x^{k+1}}{x^k} \right| = \lim_{k \to \infty} |x| = |x|$$

$$-1 < x < 1 \Leftrightarrow |x| < 1$$

$$x \leq -1 \text{ or } x \geq 1 \Leftrightarrow |x| \geq 1$$

If $\lim_{k\to\infty} \left|\frac{a_{k+1}}{a_k}\right|$ is < 1, then

$\sum_{k=0}^{\infty} a_k$ converges. If $\lim_{k\to\infty} \left|\frac{a_{k+1}}{a_k}\right| > 1$,

then $\sum_{k=0}^{\infty} a_k$ diverges. If the

limit $\lim_{k\to\infty} \left|\frac{a_{k+1}}{a_k}\right| = 1$ or it doesn't

exist, then the ratio test gives

no information.

This happens with $\sum_{k=1}^{\infty} \frac{1}{k^p}$,

for example.

But for power series, like

$\sum_{k=0}^{\infty} (3x+k)^k/(2^k)$, the ratio test

gives useful information.

$$a_k = \left(\frac{3x+k}{2}\right)^k \Rightarrow \frac{a_{k+1}}{a_k} = \frac{((3x+k+1)/2)^{k+1}}{((3x+k)/2)^k}$$

$$\frac{a_{k+1}}{a_k} = \frac{(3x+k+1)^{k+1}/2}{(3x+k)^k} = \frac{(3x+k+1)^k (3x+k+1)/2}{(3x+k)^k}$$

$$= \left(\frac{3x+k+1}{3x+k}\right)^k \cdot \frac{3x+k+1}{2}$$

$$= \left(\frac{3x+k}{3x+k} + \frac{1}{3x+k}\right)^k \frac{3x+k+1}{2}$$

$$= \left(1 + \frac{1}{3x+k}\right)^k \cdot \frac{3x+k+1}{2}$$

$$L = \lim_{k\to\infty} \left|\frac{a_{k+1}}{a_k}\right| = \lim_{k\to\infty} \frac{a_{k+1}}{a_k} = ?$$

When k is big enough,

$3x+k+1 > 0$ and

$1/(3x+k) > 0$

$$\ln\left(\frac{a_{k+1}}{a_k}\right) = \underbrace{k}_{\to\infty} \underbrace{\ln\left(1 + \frac{1}{3x+k}\right)}_{\to \ln(1+0) = 0} + \underbrace{\ln(3x+k+1)}_{\to\infty} - \ln 2$$

$(+\text{large})(+\text{small}) + \text{large} - \ln 2$

$= +\text{large}$ (when k is $+\text{large}$)

$$\frac{a_{k+1}}{a_k} = e^{\ln(a_{k+1}/a_k)} = e^{+\text{large}} = +\text{large}$$

So, the ratio test result is always ∞, which is >1, so the series

$\sum_{k=0}^{\infty} a_k$ diverges (for all x).

$$a_k = \left(\frac{3x+k}{2}\right)^k.$$

$$\left(\frac{3x+0}{2}\right)^0 + \left(\frac{3x+1}{2}\right)^1 + \left(\frac{3x+2}{2}\right)^2 + \cdots$$

diverges.

$$\sum_{k=0}^{\infty}\left[\left(\frac{3x+1}{2}\right)^k k\right] \quad \text{converges}$$

if $\left|\frac{3x+1}{2}\right| < 1$

$$\Leftrightarrow |3x+1| < 2$$

$$\Leftrightarrow -2 < 3x+1 < 2$$

$$\Leftrightarrow -3 < 3x < 1$$

$$\Leftrightarrow -1 < x < \frac{1}{3}$$

because $\lim_{k\to\infty}\left|\frac{((3x+1)/2)^{k+1}/(k+1)}{((3x+1)/2)^{k}/k}\right|$

$$= \lim_{k\to\infty}\left|\frac{(3x+1)/2}{(k+1)/k}\right| = \lim_{k\to\infty}\left|\frac{(3x+1)/2}{1+1/k}\right|$$

$$= \left|\frac{(3x+1)/2}{1+0}\right| = \left|\frac{3x+1}{2}\right|$$

If $\left|\frac{3x+1}{2}\right| > 1$, then

$\sum_{k=0}^{\infty}\left[\left(\frac{3x+1}{2}\right)^{k} k\right]$ diverges.

If $\left|\frac{3x+1}{2}\right| = 1$, then we need to use other method to determine convergence/divergence.

$$\left|\frac{3x+1}{2}\right| = 1 \Leftrightarrow \frac{3x+1}{2} = \pm 1$$

$$\sum_{k=0}^{\infty}\left(\frac{3x+1}{2}\right)^{k} k = \begin{cases} 0 + 1 + 2 + 3 + \cdots = \infty \\ \text{or} \\ 0 - 1 + 2 - 3 + \cdots \text{ diverges} \end{cases}$$

$$0 - 1 + 2 - 3 + 4 - 5 + \cdots + (-1)^k k$$

$$= \begin{cases} k/2 & \text{if } k \text{ even} \\ -(k+1)/2 & \text{if odd} \end{cases}$$

when k is large you oscillate between +large & -large.

$0 = 0$

$0 - 1 = -1$

$0 - 1 + 2 = 1$

$0 - 1 + 2 - 3 = -2$

$0 - 1 + 2 - 3 + 4 = 2$

$0 - 1 \cdots - 5 = -3$

$0 - 1 \cdots + 6 = 3$

$0 - 1 \cdots - 7 = -4$

$0 - 1 \cdots + 8 = 4$

The nice thing is that when the ratio test returns < 1, we can integrate & differentiate term by term.

$$-1 < x < 1 \Rightarrow \left(\sum_{k=0}^{\infty} x^k\right)' = \sum_{k=0}^{\infty} k x^{k-1}$$

$$\left(\frac{1}{1-x}\right)' = \frac{1}{(1-x)^2}$$

$$\int_a^b \frac{dx}{1-x} = \sum_{k=0}^{\infty} \int_a^b x^k \, dx \quad \text{when } -1 < a \le b < 1$$

$$\ln \frac{1-a}{1-b} \; ?$$

Yes: $\displaystyle\int_a^b \frac{dx}{1-x} = \int_{1-a}^{1-b} \frac{-du}{u} = -\ln|u| \Big|_{1-a}^{1-b} = \ln|u| \Big|_{1-b}^{1-a}$

$u = 1-x, \quad dx = -du$ $\quad = \ln|1-a| - \ln|1-b|$

$\ln \frac{1-a}{1-b} = \ln\left|\frac{1-a}{1-b}\right|$

$$e^x = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \frac{x^4}{4!} + \cdots = \sum_{k=0}^{\infty} \frac{x^k}{k!}$$

Ratio test returns $0 < 1$ for all x.

$$\int_0^t e^{\sin x} \, dx = \sum_{k=0}^{\infty} \int_0^t \frac{(\sin x)^k}{k!} \, dx$$

$$\int_0^t e^{1-x^3} \, dx = e^1 \int_0^t e^{-x^3} \, dx$$

$$= e \sum_{k=0}^{\infty} \int_0^t \frac{(-x^3)^k}{k!} \, dx$$

$$= e \sum_{k=0}^{\infty} \frac{(-1)^k}{k!} \int_0^t x^{3k} \, dx$$

$$= e \sum_{k=0}^{\infty} \frac{(-1)^k}{k!} \frac{x^{3k+1}}{3k+1} \Bigg|_0^t$$

$$= e \sum_{k=0}^{\infty} \frac{(-1)^k}{k!} \cdot \frac{t^{3k+1}}{3k+1}$$

$$= e\left(\frac{t}{1} - \frac{t^4}{4} + \frac{t^7}{2!\,7} - \frac{t^{10}}{3!\,10} + \cdots\right)$$

$$\begin{array}{cccc} k=0 & k=1 & k=2 & k=3 \end{array}$$

$$= \int_0^t e^{1-x^3}\,dx$$

Test: no calculator; notes OK

$$I = \int_5^\infty x e^{-x}\,dx = \lim_{t\to\infty} \int_5^t x e^{-x}\,dx$$

I.B.P.

$$\begin{array}{c|c|c|c} & x & e^{-x} & \\ d\downarrow & 1 & -e^{-x} & \downarrow\int \\ & 0 & e^{-x} & \end{array}$$

$$\oplus\; x(-e^{-x})$$

$$\ominus\; 1(e^{-x})$$

$$I = \lim_{t\to\infty} \left(-xe^{-x} - e^{-x}\right)\Big|_5^t$$

$$I = \lim_{t\to\infty} \Big(\underbrace{-te^{-t}}_{?} - \underbrace{e^{-t}}\Big) - \left(-5e^{-5} - e^{-5}\right)$$

$t = +\text{large}$ $\qquad e^{-\text{large}} = \dfrac{1}{e^{+\text{large}}} = \dfrac{1}{+\text{large}}$

+small

$$\lim_{t\to\infty} \underbrace{-t}_{-\text{large}}\,\underbrace{e^{-t}}_{+\text{small}} = \lim_{t\to\infty} \frac{\overset{-\text{large}}{-t}}{\underset{+\text{large}}{e^{t}}}$$

Use L's rule for $\pm\frac{\infty}{\infty}$ or $\frac{0}{0}$.

$$\lim_{t\to\infty} \frac{-t}{e^{t}} = \lim_{t\to\infty} \frac{(-t)'}{(e^{t})'} = \lim_{t\to\infty} \frac{-1}{e^{t}} = 0$$

$$\frac{-1}{+\text{large}} \to 0$$

$$I = (0-0) - (-5e^{-5} - e^{-5}) = 6e^{-5}$$