Estimating sums of infinite series

Past: $\begin{cases} \text{I} & \text{Alternating series} \\ \text{II} & \text{Integral estimates} \end{cases}$

Today: III Geometric series estimates

for all x, $\quad e^x = \sum_{k=0}^{\infty} \frac{x^k}{k!}$

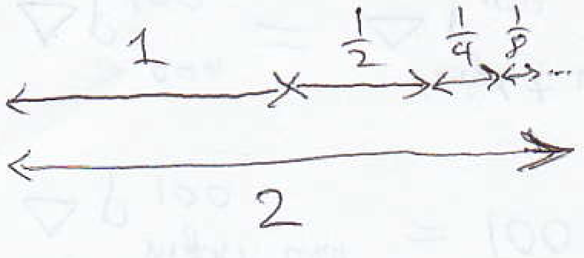
$$e = \sum_{k=0}^{\infty} \frac{1}{k!} \qquad k! = \underbrace{1 \cdot 2 \cdot 3 \cdot \cdots (k-1) \cdot k}_{\text{positive}}$$

$$e = \underbrace{\sum_{k=0}^{n} \frac{1}{k!}}_{\text{estimate}} + \underbrace{\sum_{k=n+1}^{\infty} \frac{1}{k!}}_{\text{remainder}}$$

(underestimate) $\Leftarrow$ (positive)

Geometric series $\quad \sum_{k=0}^{\infty} x^k = \frac{1}{1-x} \quad$ when $-1 < x < 1$

$$\sum_{k=0}^{\infty} \left(\tfrac{1}{2}\right)^k = 2$$



Find a constant multiple of a geometric series > the remainder.

$$\text{remainder} = \sum_{k=n+1}^{\infty} \frac{1}{k!} = \frac{1}{(n+1)!} + \frac{1}{(n+2)!} + \frac{1}{(n+3)!} + \frac{1}{(n+4)!} + \cdots$$

$$= \frac{1}{(n+1)!} \sum_{k=n+1}^{\infty} \frac{(n+1)!}{k!}$$

$$= \frac{1}{(n+1)!} \left[\underbrace{\frac{(n+1)!}{(n+1)!}}_{1} + \frac{(n+1)!}{(n+2)!} + \frac{(n+1)!}{(n+3)!} + \cdots \right]$$

$$\frac{(n+1)!}{(n+2)!} = \frac{1 \cdot 2 \cdot 3 \cdot 4 \cdot \cdots (n-1) \cdot n \cdot (n+1)}{1 \cdot 2 \cdot 3 \cdot 4 \cdot \cdots (n-1) \cdot n \cdot (n+1)(n+2)}$$

$$\frac{(n+1)!}{(n+2)!} = \frac{1}{n+2}$$

$$\frac{(n+1)!}{(n+3)!} = \frac{1 \cdot 2 \cdot \cdots \cdot (n+1)}{1 \cdot 2 \cdot \cdots \cdot (n+1)(n+2)(n+3)} = \frac{1}{(n+2)(n+3)}$$

$$\sum_{k=n+1}^{\infty} \frac{1}{k!} = \frac{1}{(n+1)!}\left[1 + \frac{1}{n+2} + \frac{1}{(n+2)(n+3)} + \frac{1}{(n+2)(n+3)(n+4)} + \frac{1}{(n+2)(n+3)(n+4)(n+5)} + \frac{1}{(n+2)\cdots(n+6)} + \cdots\right]$$

Compare to

$$\frac{1}{(n+1)!}\underbrace{\left[1 + \frac{1}{n+2} + \frac{1}{(n+2)^2} + \frac{1}{(n+2)^3} + \frac{1}{(n+2)^4} + \cdots\right]}_{\sum_{k=0}^{\infty}\left(\frac{1}{n+2}\right)^k}$$

$$\frac{1}{336} = \frac{1}{6\cdot 7\cdot 8}$$

Try $n=4$:

$$\sum_{k=n+1}^{\infty} = \frac{1}{4!}\left[1 + \overbrace{\frac{1}{(4+2)}}^{1/6} + \overbrace{\frac{1}{(4+2)(4+3)}}^{1/42} + \cdots\right]$$

$$\frac{1}{4!}\left[1 + \underbrace{\frac{1}{(4+2)}}_{1/6} + \underbrace{\frac{1}{(4+2)^2}}_{1/36} + \underbrace{\cdots}_{1/6^3 \,=\, 1/216}\right]$$

$$\text{remainder} = \sum_{k=n+1}^{\infty}\frac{1}{k!} < \frac{1}{(n+1)!}\sum_{k=0}^{\infty}\left(\frac{1}{n+2}\right)^k$$

$$\text{remainder} < \frac{1}{(n+1)!} \frac{1}{1-\frac{1}{n+2}}$$

$$\frac{1}{1-\frac{1}{n+2}} = \frac{1}{\frac{n+2}{n+2}-\frac{1}{n+2}} = \frac{1}{\frac{n+1}{n+2}} = \frac{n+2}{n+1}$$

$$\text{remainder} < \frac{1}{(n+1)!} \cdot \frac{n+2}{n+1}$$

$$\sum_{k=0}^{n} \frac{1}{k!} < e < \sum_{k=0}^{n} \frac{1}{k!} + \underbrace{\frac{1}{(n+1)!} \cdot \frac{n+2}{n+1}}$$

Try $n=4$:

$$\overbrace{\frac{1}{0!}+\frac{1}{1!}+\frac{1}{2!}+\frac{1}{3!}+\frac{1}{4!}}^{2.708333\ldots} < e < \frac{1}{0!}+\cdots+\frac{1}{4!} + \overbrace{\frac{1}{5!}\cdot\frac{6}{5}}$$

1/100

If we must find 1/5!, let's use it on the left side too.

$$\frac{1}{0!}+\frac{1}{1!}+\frac{1}{2!}+\cdots+\frac{1}{5!} < e$$

$$\underbrace{\frac{1}{1}+\frac{1}{1}+\frac{1}{2}+\frac{1}{6}+\frac{1}{24}+\frac{1}{120}}_{2.71666\ldots} < e < \underbrace{\frac{1}{1}+\cdots+\frac{1}{24}+\frac{1}{120}\cdot\frac{6}{5}}_{2.718333333\ldots}$$

You can estimate e^x for any x:

$$e^x = \sum_{k=0}^{\infty} \frac{x^k}{k!}$$

HW #1 Get lower and upper bounds for e^3 less than 0.05 apart.

#2 Use alternating series estimates to get lower and upper bounds of e^{-3} less than 0.0005 apart.