Taylor series (9.10, 9.11)

$$f(x) = \underbrace{\sum_{k=0}^{n} \frac{f^{(k)}(0)}{k!} x^k}_{\substack{\text{nth degree} \\ \text{Taylor expansion} \\ \text{of } f \text{ at } x=0.}} + \underbrace{\int_0^x (x-t)^k \frac{f^{(k+1)}(t)}{k!} \, dt}_{\text{remainder } R_n(x)}$$

$$\underbrace{\sin(2)}_{0.909297} = \underbrace{\sum_{k=0}^{7} \frac{\sin^{(k)}(0)}{k!} 2^k}_{\underbrace{\frac{2^1}{1!} - \frac{2^3}{3!} + \frac{2^5}{5!} - \frac{2^7}{7!}}_{0.907936}} + \underbrace{\int_0^2 (2-t)^7 \frac{\sin^{(8)}(t)}{7!} \, dt}_{\underbrace{0.0013609}_{R_7(2)}}$$

For $\sin(x)$, $\lim_{n \to \infty} R_n(x) \to 0$ for all x.

Therefore, $\sin(x) = \sum_{k=0}^{\infty} \frac{\sin^{(k)}(0)}{k!} x^k$

$$= x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \frac{x^9}{9!} - \cdots$$

$$= \sum_{k=0}^{\infty} (-1)^k x^{2k+1} / (2k+1)!$$

HW #1 Approximate $\sin(0.7)$ using the Taylor expansion at $x=0$. Use alternating series estimates to get lower and upper bounds $<10^{-5}$ apart.

#2 Do the same for $\sin(7)$.

$\lim_{n\to\infty} R_n(x) = 0$ for all x, for $\cos(x)$.

$$\cos(x) = \sum_{k=0}^{\infty}\left(\frac{\cos^{(k)}(0)}{k!}\right)x^k = 1 - \frac{x^2}{2} + \frac{x^4}{24} - \frac{x^6}{720} + \frac{x^8}{40320} - \frac{x^{10}}{10!} + \frac{x^{12}}{12!} - \frac{x^{14}}{14!} + \cdots$$

$$= \sum_{k=0}^{\infty}(-1)^k\frac{x^{2k}}{(2k)!}$$

k	$\cos^{(k)}(x)$	$\cos^{(k)}(0)$	$k!$	x^k
0	$\cos x$	1	1	1
1	$-\sin x$	0	1	x
2	$-\cos x$	-1	2	x^2
3	$\sin x$	0	6	x^3
4	$\cos x$	1	24	x^4
5	$-\sin x$	0	120	x^5
6	$-\cos x$	-1	720	x^6
7	$\sin x$	0	5040	x^7
8	$\cos x$	1	40320	x^8
$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$

$$(1+x)^p = 1 + px + \frac{p(p-1)}{2!}x^2 + \frac{p(p-1)(p-2)}{3!}x^3$$

$$+ \frac{p(p-1)(p-2)(p-3)}{4!}x^4 + \cdots$$ ~~because~~

for $-1 \le x \le 1$. because there $\lim_{n\to\infty} R_n = 0$.

[Binomial series] = [Taylor series of $(1+x)^p$ at $x=0$]

$f(x) = (1+x)^p$ $\quad f'(x) = p(1+x)^{p-1}$

$f'(0) = p\underbrace{(1+0)^{p-1}}_{1} = p$

$f(0) = 1$

$f''(x) = p(p-1)(1+x)^{p-2} \Rightarrow f''(0) = p(p-1)$

<u>HW #3</u> Write $\sqrt[3]{10} = \sqrt[3]{8 \cdot \frac{10}{8}} = 2\sqrt[3]{10/8}$

$= 2\sqrt[3]{1 + 2/8} = 2\left(1 + \frac{1}{4}\right)^{1/3}$ and

estimate this to 3 decimal places using the Binomial series and can ~~geometric~~ alternating series estimate to get lower & upper bounds.