Classically, the kinetic energy of a particle with mass $m > 0$ and speed v is $\frac{1}{2}mv^2$. Then along came Einstein.

Always: $m > 0 \Rightarrow 0 \le v < c = 3.00 \times 10^8$ m/s

and energy is not $\frac{1}{2}mv^2$; it's

$$\underbrace{\left(\frac{1}{\sqrt{1-v^2/c^2}} - 1\right)}_{\text{approximate with Binomial series}} mc^2.$$

$\frac{1}{2}mv^2$ is approx. correct when $v \ll c$.

Notation: $\beta = \frac{v}{c}$ $\quad \gamma = \frac{1}{\sqrt{1-\beta^2}} = \frac{1}{\sqrt{1-v^2/c^2}}$

$$0 \le v < c \Rightarrow \frac{0}{c} \le \frac{v}{c} < \frac{c}{c} \Rightarrow 0 \le \beta < 1$$

Classical $E = \frac{1}{2}mv^2 = \frac{1}{2}m\frac{v^2}{c^2}c^2 = \boxed{\frac{1}{2}m\beta^2c^2}$

Einstein's special relativity: $E = \boxed{(\gamma - 1)mc^2}$

classical: $E/(mc^2) = \frac{1}{2}\beta^2$

relativistic: $E/(mc^2) = \gamma - 1 = \frac{1}{\sqrt{1-\beta^2}} - 1$

When β is small, is $\frac{1}{2}\beta^2 \approx \frac{1}{\sqrt{1-\beta^2}} - 1$?

$$(1+x)^p = 1 + px + p(p-1)x^2/2!$$
$$+ p(p-1)(p-2)\,x^3/3!$$
$$+ p(p-1)(p-2)\,x^4/4! + \cdots$$

when $\boxed{-1 < x < 1}$

$$\frac{1}{\sqrt{1-\beta^2}} = (1-\beta^2)^{-\frac{1}{2}} \qquad p = -\frac{1}{2} \qquad x = -\beta^2$$

✓

$$0 \le \beta < 1 \Rightarrow 0^2 \le \beta^2 < 1^2 \Rightarrow 0 \le \beta^2 < 1$$

$$\Rightarrow -0 \ge -\beta^2 > -1 \Rightarrow \boxed{1 > 0 \ge \underbrace{-\beta^2}_{x} > -1}$$

$$(1-\beta^2)^{-\frac{1}{2}} = 1 + (-\tfrac{1}{2})(-\beta^2) + (-\tfrac{1}{2})(-\tfrac{3}{2})(-\beta^2)^2/2! + (-\tfrac{1}{2})(-\tfrac{3}{2})(-\tfrac{5}{2})(-\beta^2)^3/3! + \cdots$$

$$(1+x)^p = 1 + px + p(p-1)x^2/2! + p(p-1)(p-2)x^3/3! + \cdots$$

$$\underbrace{(1-\beta^2)^{-\frac{1}{2}}}_{\frac{1}{\sqrt{1-\beta^2}}} = 1 + \frac{1}{2}\beta^2 + \left(\frac{1}{2}\right)\left(\frac{3}{2}\right)\frac{\beta^4}{1\cdot 2} + \left(\frac{1}{2}\right)\left(\frac{3}{2}\right)\left(\frac{5}{2}\right)\frac{\beta^6}{1\cdot 2\cdot 3}$$
$$+ \left(\frac{1}{2}\right)\left(\frac{3}{2}\right)\left(\frac{5}{2}\right)\left(\frac{7}{2}\right)\frac{\beta^8}{1\cdot 2\cdot 3\cdot 4} + \left(\frac{1}{2}\right)\cdots\left(\frac{9}{2}\right)\frac{\beta^{10}}{1\cdot 2\cdot 3\cdot 4\cdot 5}$$
$$+ \cdots$$

$$\frac{1}{\sqrt{1-\beta^2}} - 1 = \frac{1}{2}\beta^2 + \left(\frac{1}{2}\right)\left(\frac{3}{2}\right)\frac{\beta^4}{1\cdot 2} + \left(\frac{1}{2}\right)\left(\frac{3}{2}\right)\left(\frac{5}{2}\right)\frac{\beta^6}{1\cdot 2\cdot 3} + \cdots$$

Is $\frac{1}{\sqrt{1-\beta^2}} - 1 \approx \frac{1}{2}\beta^2$ when β is ≈ 0?

We need to estimate how big or small the remaining terms add up to be:

$$\frac{1\cdot 3\cdot \beta^4}{2\cdot 4} + \frac{1\cdot 3\cdot 5\cdot \beta^6}{2\cdot 4\cdot 6} + \frac{1\cdot 3\cdot 5\cdot 7\,\beta^8}{2\cdot 4\cdot 6\cdot 8} + \cdots$$

[Calculator failed to accurately compute $1/\sqrt{1-\beta^2} - 1$ for very small β.]

We'll estimate this next:

$$\frac{1\cdot 3\cdot \beta^4}{2\cdot 4}\left(1 + \frac{5}{6}\beta^2 + \frac{5\cdot 7}{6\cdot 8}\beta^4 + \frac{5\cdot 7\cdot 9}{6\cdot 8\cdot 10}\beta^6 + \cdots\right)$$

$$< \frac{1\cdot 3\cdot \beta^4}{2\cdot 4}\left(1 + \beta^2 + \beta^4 + \beta^6 + \cdots\right)$$

$$= \frac{1\cdot 3\,\beta^4}{2\cdot 4}\underbrace{\left(1 + \beta^2 + (\beta^2)^2 + (\beta^2)^3 + \cdots\right)}_{= 1/(1-\beta^2)}$$

For the small β we're interested in, certainly $\beta < \frac{1}{2}$ (for example), so $1/(1-\beta^2) \le 1/(1-1/4) = 4/3$

So, our remainder $\frac{1\cdot 3\cdot \beta^4}{2\cdot 4} + \frac{1\cdot 3\cdot 5\,\beta^6}{2\cdot 4\cdot 6} + \cdots$

is $< \frac{1\cdot 3\beta^4}{2\cdot 4}\cdot\frac{4}{3} = \frac{1}{2}\beta^4$

Conclusion: $\frac{1}{\sqrt{1-\beta^2}} - 1 = \frac{1}{2}\beta^2 + \text{remainder}$

and $0 \le \text{remainder} < \frac{1}{2}\beta^4$,

so $\frac{1}{2}\beta^2 \le \frac{1}{\sqrt{1-\beta^2}} - 1 < \frac{1}{2}\beta^2 + \frac{1}{2}\beta^4$

So, if $\beta \approx 0$, then $\beta^4 \approx 0$, so

$\frac{1}{2}\beta^2 + \frac{1}{2}\beta^4 \approx \frac{1}{2}\beta^2$ (and β^4 will be much smaller than β^2),

so $\frac{1}{2}\beta^2 \approx \frac{1}{\sqrt{1-\beta^2}} - 1$.

Example: $\beta = 10^{-8}$:

$\frac{1}{2}\beta^2 = \frac{1}{2}\cdot 10^{-16} = 5\cdot 10^{-17}$

$\frac{1}{2}\beta^4 = \frac{1}{2}\cdot 10^{-32} = 5\cdot 10^{-33}$

$$\underbrace{\frac{1}{2}\beta^2}_{5\cdot 10^{-17}} < \frac{1}{\sqrt{1-\beta^2}} - 1 < \underbrace{\frac{1}{2}\beta^2 + \frac{1}{2\cancel{8}}\beta^4}_{5\cdot 10^{-17} + 5\cdot 10^{-33}}$$

$$0,\underbrace{00\ldots0}_{16}5 \qquad 0.\underbrace{0\ldots0}_{16}5\underbrace{0\ldots0}_{15}5$$

A very good estimate!

HW: Use the binomial series to get a similar estimate of γ when $\beta = 1 - 10^{-15}$.