ratio test
integral test ← history

T6

alternating series test
alternating series estimates
integral estimates
geometric series estimates
Taylor series (sine, cosine, binomial)

Comprehensive T7 (final); emphasis on topics from after T6

Ratio test: If $\lim_{k\to\infty}\left|\frac{a_{k+1}}{a_k}\right| < 1$,

then the series converges and you can estimate it using geometric series estimates.

If $\lim_{k\to\infty}\left|\frac{a_{k+1}}{a_k}\right| > 1$, the series diverges.

If $\lim_{k\to\infty}\left|\frac{a_{k+1}}{a_k}\right| = 1$ or does not exist, then try alternating series test or the integral test; there won't be a geometric series estimate, but there might be an integral estimate or an alternating series estimate.

IF the series is alternating and $\lim_{k\to\infty}\left|\frac{a_{k+1}}{a_k}\right| < 1$, use the simpler alternating series estimate.

$$\hookrightarrow e^{-1} = \sum_{k=0}^{\infty} \frac{(-1)^k}{k!}$$

Example where series is <u>not</u> alternating and $\lim_{k\to\infty}\left|\frac{a_{k+1}}{a_k}\right| < 1$:

$$e = e^1 = \sum_{k=0}^{\infty} \frac{1}{k!}$$ ← Need geometric series estimate.

Example where $\lim_{k\to\infty}\left|\frac{a_{k+1}}{a_k}\right| = 1$:

$$\sum_{k=1}^{\infty} \frac{-(-1)^k}{k} = \frac{1}{1} - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \frac{1}{5} - \frac{1}{6} \cdots$$

↳ This converges by the A.S.T. & you can estimate w/ alt. ser. est.

Another example where $\lim_{k\to\infty}\left|\frac{a_{k+1}}{a_k}\right| = 1$:

$$\sum_{k=1}^{\infty} \frac{1}{k^6} = \frac{1}{1^6} + \frac{1}{2^6} + \frac{1}{3^6} + \frac{1}{4^6} + \cdots$$

$\hookrightarrow$ converges by the integral test & you can estimate w/ ~~the~~ an ~~an~~ integral estimate.

Another: $\sum_{k=1}^{\infty} \frac{1}{k}$ $\quad \lim_{k\to\infty}\left|\frac{a_{k+1}}{a_k}\right| = 1$

$\leftarrow a_k = \frac{1}{k}$

series diverges by the integral test.

$$\sum_{k=1}^{\infty} \underbrace{\left(\frac{3k+5}{\sqrt{k^2+1}}\right)^k} \Rightarrow 3 = \lim_{k\to\infty}\left|\frac{a_{k+1}}{a_k}\right| > 1$$

$\hookrightarrow \sim 3^k$ when k large

$$\frac{3^{k+1}}{3^k} = 3$$

$\to$ The series diverges.

$$\sum_{k=1}^{\infty} \underbrace{(-1)^k \left(\frac{1+k^2}{2k^2}\right)^3} \qquad \lim_{k\to\infty} \left|\frac{a_{k+1}}{a_k}\right| = 1$$

$\downarrow$ for k large

$a_k \sim \pm\frac{1}{8}$, not 0, so

the series diverges by the alt. ser. test

$$- \; - \; - \; - \; - \sim\tfrac{1}{8} \uparrow \; \downarrow \; \uparrow \; \downarrow \; \uparrow \; \downarrow \; \cdots$$

k large

$$\frac{1}{1-x} = 1 + x + x^2 + x^3 + \cdots \quad \text{for } |x| < 1$$

$$e^x = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \cdots \text{ for all } x$$

$$\cos x = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} + \cdots \text{ for all } x$$

$$\sin x = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \cdots \text{ for all } x$$

$$(1+x)^p = 1 + px + \frac{p(p-1)}{2!}x^2 + \frac{p(p-1)(p-2)}{3!}x^3 + \cdots$$

for $|x| < 1$; for all p.

You can get more series by integrating, differentiating, substituting

$$\arcsin x = \int_0^x \frac{dt}{\sqrt{1-t^2}} = \int_0^x \underbrace{(1-t^2)^{-\frac{1}{2}}}_{\text{binomial series}} dt$$

$$\underbrace{1}_{\int_0^x 1\,dt} + \underbrace{\left(-\tfrac{1}{2}\right)(-t^2)}_{\int_0^x (-\frac{1}{2})(-t^2)\,dt} + \underbrace{\frac{\left(-\frac{1}{2}\right)\left(-\frac{1}{2}-1\right)}{2!}(-t^2)^2}_{\int_0^x (-\frac{1}{2})(-\frac{1}{2}-1)(-t^2)^2\,dt} + \cdots$$

Integrate each term.

Final exam May 12, 2PM, KWAC 129

notes + calculator OK.

Taylor series expansion at $x=0$:

$$f(x) = f(0) + f'(0)x + f''(0)\frac{x^2}{2!} + f'''(0)\frac{x^3}{3!} + f^{(4)}(0)\frac{x^4}{4!} + \cdots$$

$$\cdots + f^{(n)}(0)\frac{x^n}{n!} + R_n$$

$$R_n = \int_0^x (x-t)^n \frac{f^{(n+1)}(t)}{n!}\,dt$$

Example: $\tan x$. $f(x) = \tan x$

k	$f^{(k)}(x)$	$f^{(k)}(0)$	$x^k/k!$
0	$\tan x$	0	1
1	$\sec^2 x$	1	x
2	$2\sec^2 x \tan x$	0	$x^2/2$
3	$2\sec^4 x + 4\sec^2 x \tan^2 x$	2	$x^3/6$

$\tan x = x + \frac{x^3}{3} + R_3$

for $x \approx 0$, $R_3 \approx 0$.

In fact R_3 is there small compared to x^3.

$$d(\sec^2 x) = 2\sec x \, d(\sec x) = 2\sec x \sec x \tan x$$

$$2d(\sec^2 x \tan x)$$

$$= 2d(\sec^2 x)\tan x + 2\sec^2 x \, d(\tan x)$$

$$= 4\sec^2 x \tan^2 x + 2\sec^4 x$$