INTEGRATING AND DIFFERENTIATING SERIES OF POWERS OF FUNCTIONS

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Let $g(x)$ be such that $g'(x)$ exists and is continuous; let $c_0, c_1, c_2, \ldots$ be a sequence of constants. Suppose that $\lim\limits_{k\to\infty}\left|\frac{c_{k+1}(g(x))^{k+1}}{c_k(g(x))^k}\right|$ exists and is < 1 for all x in some interval $[a, b]$, except where $g(x) = 0$. By the ratio test, $\sum\limits_{k=0}^{\infty} c_k(g(x))^k$ converges for all x in $[a, b]$. Call this sum $f(x)$.

We would hope that

$$\int_a^b f(x)\,dx = \sum_{k=0}^{\infty}\left(c_k \int_a^b g(x)^k\,dx\right)$$

and, when $a < x < b$,

$$\frac{df}{dx} = \sum_{k=0}^{\infty}\left(c_k \frac{d(g(x)^k)}{dx}\right),$$

but hope is not a proof, and, despite how natural they look, these equations are not obviously true.

For example, it can be shown that $\sum\limits_{k=1}^{\infty} \frac{\sin kx}{k}$ is discontinuous (and therefore not differentiable) at the points $x = 0, \pm 2\pi, \pm 4\pi, \ldots$. (At these x-valus, the value of the sum is 0, the limit from the left is $-\pi/2$, and the limit from the right is $+\pi/2$. Plot $\sum\limits_{k=1}^{10} \frac{\sin kx}{k}$ to see an approximation of these discontinuities.) These discontinuities occur even though the series converges for all x, and $\frac{\sin kx}{k}$ can differentiated (with respect to x) as many times as we like.

The purpose of this note is to prove that, under our assumptions listed in the first paragraph, we can safely integrate and differentiate $f(x) = \sum\limits_{k=0}^{\infty} c_k g(x)^k$ term by term.

Theorem 1. *Let $g(x)$ be continuous on $[a, b]$; let $c_0, c_1, c_2, \ldots$ be a sequence of constants. Suppose that $\lim\limits_{k\to\infty}\left|\frac{c_{k+1}g(x)^{k+1}}{c_k g(x)^k}\right|$ exists and is < 1 for all x in $[a, b]$, except where $g(x) = 0$. It follows that*

$$\int_a^b \left(\sum_{k=0}^{\infty} c_k g(x)^k\right) dx = \sum_{k=0}^{\infty}\left(\int_a^b c_k g(x)^k\,dx\right).$$

Proof. Let $f(x) = \sum\limits_{k=0}^{\infty} c_k g(x)^k$, which, by the ratio test, converges for all x in $[a, b]$. Before we can talk about what $\int\limits_a^b f(x)\,dx$ equals, we need to show that it exists. To show that this integral exists, it is sufficient (see section 4.2 of Keisler's book [1]) to show that $f(x)$ is continuous on $[a, b]$.

To show continuity, we must show that if u, v are in $[a, b]$ and $u \approx v$, then $f(u) \approx f(v)$. For any finite n, $\sum\limits_{k=0}^{n} c_k g(x)^k$ is just a finite sum of constant multiples of $g(x)$, so it is continuous on $[a, b]$, so

$\sum_{k=0}^{n} c_k g(u)^k \approx \sum_{k=0}^{n} c_k g(v)^k$. We can break up $f(x)$ into two sums: $f(x) = \sum_{k=0}^{n} c_k g(x)^k + \sum_{k=n+1}^{\infty} c_k g(x)^k$. Therefore,

$$f(u) - f(v) = \sum_{k=0}^{n} c_k g(u)^k - \sum_{k=0}^{n} c_k g(v)^k + \sum_{k=n+1}^{\infty} c_k g(u)^k - \sum_{k=n+1}^{\infty} c_k g(v)^k,$$

which, since $\sum_{k=0}^{n} c_k g(u)^k \approx \sum_{k=0}^{n} c_k g(v)^k$, implies

$$f(u) - f(v) \approx \sum_{k=n+1}^{\infty} c_k g(u)^k - \sum_{k=n+1}^{\infty} c_k g(v)^k.$$

Therefore, no matter what our choice of n, $\sum_{k=n+1}^{\infty} c_k g(u)^k - \sum_{k=n+1}^{\infty} c_k g(v)^k$ is infinitely close to the same real, the standard part of $f(u) - f(v)$; call this real s. We will next show that $s = 0$, which implies that $f(u) - f(v) \approx 0$, from which our desired claim $f(u) \approx f(v)$ follows immediately.

Let us use our assumption that $\lim_{k\to\infty} \left| \frac{c_{k+1} g(x)^{k+1}}{c_k g(x)^k} \right|$ exists and is < 1 for all x in $[a, b]$ where $g(x) \neq 0$. The theorem is trivially true is $g(x) = 0$ on all of $[a, b]$, so we may assume that $g(x_1) \neq 0$ for some x_1 in $[a, b]$. First, notice that $\lim_{k\to\infty} \left| \frac{c_{k+1} g(x_1)^{k+1}}{c_k g(x_1)^k} \right| = |g(x_1)| \lim_{k\to\infty} \left| \frac{c_{k+1}}{c_k} \right|$, so $\lim_{k\to\infty} \left| \frac{c_{k+1}}{c_k} \right|$ exists; call this limit L. We now have $|g(x)|L < 1$ for all x in $[a, b]$. Since $g(x)$ is continuous on $[a, b]$ and $|x|$ is continuous everywhere, $|g(x)|$ is continuous. Therefore, by the Extreme Value Theorem (section 3.8 of [1]), there is some real x_0 in $[a, b]$ such that $|g(x_0)| \geq |g(x)|$ for all x in $[a, b]$; let $M = |g(x_0)|$. It follows that $ML = |g(x_0)|L < 1$; choose a real p such that $ML < p < 1$. (One possible choice is the midpoint $(ML + 1)/2$.) We then have $L < p/M$. When H is an infinite positive hyperinteger, $\left| \frac{c_{H+1}}{c_H} \right| \approx L$, so $\left| \frac{c_{H+1}}{c_H} \right| < p/M$. Every $n \geq H$ is also infinite, so $\left| \frac{c_{m+1}}{c_m} \right| < p/M$ for all $m \geq H$. By the Transfer Principle, there is a standard (finite) integer h such that $\left| \frac{c_{m+1}}{c_m} \right| < p/M$ for all $m \geq h$. Therefore, for all $k \geq h$, we have

$$|c_k| = \left| c_h \frac{c_{h+1}}{c_h} \frac{c_{h+2}}{c_{h+1}} \frac{c_{h+3}}{c_{h+2}} \cdots \frac{c_k}{c_{k-1}} \right| < |c_h| \left(\frac{p}{M} \right)^{k-h},$$

which implies $-|c_h| \left(\frac{p}{M}\right)^{k-h} < c_k < |c_h| \left(\frac{p}{M}\right)^{k-h}$. Therefore, whenever $n + 1 \geq h$ and x is in $[a, b]$, we have

$$-|c_h| \sum_{k=n+1}^{\infty} \left(\frac{p}{M} \right)^{k-h} g(x)^k < \sum_{k=n+1}^{\infty} c_k g(x)^k < |c_h| \sum_{k=n+1}^{\infty} \left(\frac{p}{M} \right)^{k-h} g(x)^k.$$

Since $|g(x)| \leq M$, we have $-M < g(x) < M$, so

$$-|c_h| \sum_{k=n+1}^{\infty} \left(\frac{p}{M} \right)^{k-h} M^k < \sum_{k=n+1}^{\infty} c_k g(x)^k < |c_h| \sum_{k=n+1}^{\infty} \left(\frac{p}{M} \right)^{k-h} M^k.$$

To ultimately get geometric series, we substitute $w = k - n - 1$ (which implies $k = w + n + 1$) into the left and right series:

$$-|c_h| \sum_{w=0}^{\infty} \left(\frac{p}{M} \right)^{w+n+1-h} M^{w+n+1} < \sum_{k=n+1}^{\infty} c_k g(x)^k < |c_h| \sum_{w=0}^{\infty} \left(\frac{p}{M} \right)^{w+n+1-h} M^{w+n+1}.$$

Simplifying, geometric series appear:

$$-|c_h|p^{n+1-h}M^h\sum_{w=0}^{\infty}p^w < \sum_{k=n+1}^{\infty}c_k g(x)^k < |c_h|p^{n+1-h}M^h\sum_{w=0}^{\infty}p^w.$$

Evaluating these geometric series, we have

$$-|c_h|\frac{p^{n+1-h}M^h}{1-p} < \sum_{k=n+1}^{\infty}c_k g(x)^k < |c_h|\frac{p^{n+1-h}M^h}{1-p}.$$

Since the above is true for all x in $[a,b]$, it is true if we replace x by u or v:

$$-|c_h|\frac{p^{n+1-h}M^h}{1-p} < \sum_{k=n+1}^{\infty}c_k g(u)^k < |c_h|\frac{p^{n+1-h}M^h}{1-p}$$

$$-|c_h|\frac{p^{n+1-h}M^h}{1-p} < \sum_{k=n+1}^{\infty}c_k g(v)^k < |c_h|\frac{p^{n+1-h}M^h}{1-p}$$

Multiplying the bottom line by -1, we get

$$|c_h|\frac{p^{n+1-h}M^h}{1-p} > -\sum_{k=n+1}^{\infty}c_k g(v)^k > -|c_h|\frac{p^{n+1-h}M^h}{1-p},$$

which we can rewrite as

$$-|c_h|\frac{p^{n+1-h}M^h}{1-p} < -\sum_{k=n+1}^{\infty}c_k g(v)^k < |c_h|\frac{p^{n+1-h}M^h}{1-p},$$

Therefore,

$$-|c_h|\frac{p^{n+1-h}M^h}{1-p} < \sum_{k=n+1}^{\infty}c_k g(u)^k < |c_h|\frac{p^{n+1-h}M^h}{1-p}$$

$$-|c_h|\frac{p^{n+1-h}M^h}{1-p} < -\sum_{k=n+1}^{\infty}c_k g(v)^k < |c_h|\frac{p^{n+1-h}M^h}{1-p}$$

Adding the last two lines,

$$-2|c_h|\frac{p^{n+1-h}M^h}{1-p} < \sum_{k=n+1}^{\infty}c_k g(u)^k - \sum_{k=n+1}^{\infty}c_k g(v)^k < 2|c_h|\frac{p^{n+1-h}M^h}{1-p}.$$

The expression $2|c_h|\frac{p^{n+1-h}M^h}{1-p}$ is just a constant multiple of p^n, and as $\lim\limits_{n\to\infty}p^n = 0$ because $0 \le p < 1$. Hence, $\lim\limits_{n\to\infty}2|c_h|\frac{p^{n+1-h}M^h}{1-p} = 0$. Therefore, for every positive real r, as n increases, $\sum\limits_{k=n+1}^{\infty}c_k g(u)^k - \sum\limits_{k=n+1}^{\infty}c_k g(v)^k$ is strictly between $-r$ and r. Since $s \approx \sum\limits_{k=n+1}^{\infty}c_k g(u)^k - \sum\limits_{k=n+1}^{\infty}c_k g(v)^k$ for all n, it follows that $-r < s < r$ for all positive reals r. Since s is real, $s = 0$, as we set out to show. Thus, f is continuous on $[a,b]$, and so $\int\limits_a^b f(x)\,dx$ exists.

Now we must show that $\int\limits_a^b f(x)\,dx = \sum\limits_{k=0}^{\infty}\left(c_k\int\limits_a^b g(x)^k\,dx\right)$. This is equation is equivalent to $\int\limits_a^b f(x)\,dx - \sum\limits_{k=0}^{\infty}\left(c_k\int\limits_a^b g(x)^k\,dx\right) = 0$. Since $\sum\limits_{k=0}^{\infty}\left(c_k\int\limits_a^b g(x)^k\,dx\right) = \lim\limits_{n\to\infty}\sum\limits_{k=0}^{n}\left(c_k\int\limits_a^b g(x)^k\,dx\right)$, we just need to show that $\lim_{n\to\infty}\left[\int\limits_a^b f(x)\,dx - \sum\limits_{k=0}^{n}\left(c_k \int\limits_a^b g(x)^k\,dx\right)\right] = 0$. Inside this last limit, for each n, we just have finitely many integrals being added and subtracted, so we can rewrite the limit as $\lim_{n\to\infty}\int\limits_a^b \left[f(x) - \sum\limits_{k=0}^{n} c_k g(x)^k\right] dx$. Since $f(x) = \sum\limits_{k=0}^{\infty} c_k g(x)^k$, the limit is $\lim_{n\to\infty}\int\limits_a^b \left[\sum\limits_{k=n+1}^{\infty} c_k g(x)^k\right] dx$. We have already shown that when $n+1 \ge h$,

$$-|c_h|\frac{p^{n+1-h}M^h}{1-p} < \sum_{k=n+1}^{\infty} c_k g(x)^k < |c_h|\frac{p^{n+1-h}M^h}{1-p}.$$

Therefore,

$$-|c_h|\frac{p^{n+1-h}M^h}{1-p}(b-a) < \int_a^b \left[\sum_{k=n+1}^{\infty} c_k g(x)^k\right] dx < |c_h|\frac{p^{n+1-h}M^h}{1-p}(b-a).$$

As $n \to \infty$, the left and right sides $\to 0$ because $p^n \to 0$. Therefore, $\lim_{n\to\infty}\int\limits_a^b \left[\sum\limits_{k=n+1}^{\infty} c_k g(x)^k\right] dx = 0$, completing the proof. $\square$

Theorem 2. *Let $g(x)$ be such that $g'(x)$ exists and is continuous on an open interval I; let $c_0, c_1, c_2, \ldots$ be a sequence of constants. Suppose that $\lim_{k\to\infty}\left|\frac{c_{k+1}(g(x))^{k+1}}{c_k(g(x))^k}\right|$ exists and is < 1 for all x in I. It follows that, for all x in I, except where $g(x) = 0$,*

$$\frac{d}{dx}\sum_{k=0}^{\infty} c_k g(x)^k = \sum_{k=0}^{\infty} c_k \frac{d(g(x)^k)}{dx}.$$

Moreover, the ratio-test limit is the same before and after differentiation:

$$\lim_{k\to\infty}\left|\frac{c_{k+1}d(g(x)^{k+1})/dx}{c_k d(g(x)^k)/dx}\right| = \lim_{k\to\infty}\left|\frac{c_{k+1}(g(x))^{k+1}}{c_k(g(x))^k}\right| < 1,$$

except where $g(x) = 0$ or $g'(x) = 0$.

Proof. Let $f(x) = \sum\limits_{k=0}^{\infty} c_k g(x)^k$, which, by the ratio test, converges for all x in $[a,b]$. Let $S(x) = \sum\limits_{k=0}^{\infty} c_k \frac{d(g(x)^k)}{dx}$; we must prove that $S(x)$ converges and that $f'(x) = S(x)$, for all x in I.

By the chain rule,

$$S(x) = \sum_{k=0}^{\infty} c_k \frac{kg(x)^{k-1}d(g(x))}{dx} = \sum_{k=0}^{\infty} kc_k g(x)^{k-1}g'(x).$$

If $a_k = c_k g(x)^k$, $b_k = d(a_k)/dx = kc_k g(x)^{k-1}g'(x)$, $g(x) \neq 0$, and $g'(x) \neq 0$, then

$$\left|\frac{b_{k+1}}{b_k}\right| = \left|\frac{(k+1)c_{k+1}g(x)^k g'(x)}{kc_k g(x)^{k-1}g'(x)}\right| = \frac{k+1}{k}\left|\frac{c_{k+1}g(x)^k}{c_k g(x)^{k-1}}\right| = \frac{k+1}{k}\left|\frac{c_{k+1}g(x)^{k+1}}{c_k g(x)^k}\right| = \left(1+\frac{1}{k}\right)\left|\frac{a_{k+1}}{a_k}\right|.$$

Therefore,

$$\lim_{k\to\infty}\left|\frac{b_{k+1}}{b_k}\right| = \lim_{k\to\infty}\left(1+\frac{1}{k}\right)\lim_{k\to\infty}\left|\frac{a_{k+1}}{a_k}\right| = 1\cdot\lim_{k\to\infty}\left|\frac{a_{k+1}}{a_k}\right| < 1.$$

Thus, the ratio-test limit is not changed by the term-by-term differentiation, except where $g(x) = 0$ or $g'(x) = 0$. Since the ratio-test limit $\lim_{k\to\infty}\left|\frac{a_{k+1}}{a_k}\right|$ for for $f(x)$ is < 1 for all x in I where $g(x) \neq 0$, the ratio-test limit $\lim_{k\to\infty}\left|\frac{b_{k+1}}{b_k}\right|$ for for $S(x)$ is < 1 for all x in I where $g(x) \neq 0$ and $g'(x) \neq 0$; where $g(x) = 0$ or $g'(x) = 0$, $0 = b_2 = b_3 = b_4 = b_5 = \cdots$. Hence, $S(x)$ converges for all x in I.

It remains to show that $f'(x)$ exists and equals $S(x)$ for all x in I. Fix a in I. For all x in I, we have $S(x) = \frac{d}{dx}\int_a^x S(t)\,dt$ (section 4.2, [1]), if $\int_a^x S(t)\,dt$ exists for all x in I. So, it suffices to show that $\int_a^x S(t)\,dt$ exists and equals $f(x) - f(a)$ for all x in I, for this implies $S(x) = \frac{d}{dx}(f(x) - f(a)) = f'(x)$.

We can express b_k as $kc_k(p(x)^k - q(x)^k)$ where p and q are the kth roots of the "positive and negative parts" of $g^{k-1}g'$, respectively. More precisely, $p(x) = \sqrt[k]{\max(0, g(x)^{k-1}g'(x))}$ and $q(x) = \sqrt[k]{\max(0, -g(x)^{k-1}g'(x))}$. The maximum of any two continuous functions is continuous (the proof is left as an exercise for the reader), and the two maxima defined above are always nonnegative, so p and q are continuous on I. For any value of x, $g(x)^{k-1}g'(x) \geq 0$, in which case $q(x) = 0$, or $g(x)^{k-1}g'(x) \leq 0$, in which case $p(x) = 0$. Therefore, for all x in I, either $S(x) = \sum_{k=0}^{\infty} b_k = \sum_{k=0}^{\infty} kc_kp(x)^k$ and $q(x) = 0$, or $S(x) = \sum_{k=0}^{\infty} b_k = \sum_{k=0}^{\infty} -kc_kq(x)^k$ and $p(x) = 0$.

Hence, for all x in I, either the ratio-test limit for $\sum_{k=0}^{\infty} kc_kp(x)^k$ is < 1, or $p(x) = 0$. Therefore, Theorem 1 applies to $\sum_{k=0}^{\infty} kc_kp(t)^k$. Likewise, Theorem 1 applies to $\sum_{k=0}^{\infty} -kc_kq(t)^k$. So, applying Theorem 1 in the second equation below,

$$
\begin{aligned}
\int_a^x S(t)\,dt &= \int_a^x \left(\sum_{k=0}^{\infty} kc_kp(t)^k\right) dt + \int_a^x \left(\sum_{k=0}^{\infty} -kc_kq(t)^k\right) dt \\
&= \sum_{k=0}^{\infty}\left(\int_a^x kc_kp(t)^k\,dt\right) + \sum_{k=0}^{\infty}\left(\int_a^x -kc_kq(t)^k\,dt\right) \\
&= \sum_{k=0}^{\infty}\left(\int_a^x kc_kp(t)^k\,dt + \int_a^x -kc_kq(t)^k\,dt\right) \\
&= \sum_{k=0}^{\infty}\left(c_k\int_a^x k(p(t)^k - q(t)^k)\,dt\right) \\
&= \sum_{k=0}^{\infty}\left(c_k\int_a^x kg(t)^{k-1}g'(t)\,dt\right) = \sum_{k=0}^{\infty}\left(c_k\int_{g(a)}^{g(x)} ku^{k-1}\,du\right) \\
&= \sum_{k=0}^{\infty} c_k\left(u^k\Big|_{g(a)}^{g(x)}\right) = \sum_{k=0}^{\infty} c_k(g(x)^k - g(a)^k) = \sum_{k=0}^{\infty} c_kg(x)^k - \sum_{k=0}^{\infty} c_kg(a)^k \\
&= f(x) - f(a).
\end{aligned}
$$

Thus, $\int_a^x S(t)\,dt = f(x) - f(a)$ as desired; the proof is complete. □

References

[1] Keisler, H. Jerome, *Elementary Calculus, an Infinitesimal Approach*, http://www.math.wisc.edu/∼keisler/calc.html, revised December 2010.