Proof of remainder formula:

First, $\int_0^x f'(t)\,dt = f(x)-f(0)$,

so $f(x) = \boxed{f(0)+\int_0^x f'(t)\,dt}$. (*)

(Same: $f(x)=f(0)+\int_0^x f^{(1)}(t)\frac{(x-t)^0}{0!}\,dt$.)

Now integrate by parts:

$$\int \underbrace{f'(t)}_{u}\,\underbrace{dt}_{dV} = uv-\int v\,du = f'(t)t - \int t f''(t)\,dt$$

$$\int_0^x f'(t)\,dt = f'(t)t\Big|_{t=0}^{t=x} - \int_0^x t f''(t)\,dt$$

$$\int_0^x f'(t)\,dt = f'(x)x - \underbrace{f'(0)\cdot 0}_{0} - \int_0^x t f''(t)\,dt$$

Now apply (*) to $f'(x)$:

$f'(x) = f'(0)+\int_0^x f''(t)\,dt$, so

$$\int_0^x f'(t)\,dt = \left[f'(0)+\int_0^x f''(t)\,dt\right]x - \int_0^x t f''(t)\,dt$$

$$\int_0^x f'(t)\,dt = f'(0)x + \int_0^x (x-t) f''(t)\,dt$$

x is independent of t, so we can pull it inside the integral as a constant multiple.

(*) again:

$$f(x) = f(0)+\int_0^x f'(t)\,dt = \boxed{f(0)+f'(0)x+\underbrace{\int_0^x (x-t)f''(t)\,dt}_{=\int_0^x f^{(2)}(t)\frac{(x-t)^1}{1!}\,dt}}$$

Integrate by parts again:

$$\int \underbrace{f''(t)}_{u}\,\underbrace{(x-t)\,dt}_{dv} = uv - \int v\,du$$

pick for v $(c=0)$

$$\int (x-t)\,dt = \int w\,(-dw) = -\frac{1}{2}w^2 + c = \overbrace{-\frac{(x-t)^2}{2}} + c$$

$w = x-t \Rightarrow dw = -dt$ because x is held constant in the integral: it's "dt," not "dx" in the integral, and x & t have no equations relating them, so changing t does not change x.

$$\rightarrow uv - \int v\,du = \underbrace{-f''(t)\frac{(x-t)^2}{2}}_{uv} + \int \frac{(x-t)^2}{2} f^{(3)}(t)\,dt$$

(from last page)

$$f(x) = f(0) + f'(0)x + \int_0^x f''(t)(x-t)\,dt$$

$$f(x) = f(0) + f'(0)x - \left(f''(t)\frac{(x-t)^2}{2}\right)\Big|_{t=0}^{t=x} + \int_0^x \frac{(x-t)^2}{2} f^{(3)}(t)\,dt$$

$$f(x) = f(0) + f'(0)x - \underbrace{f''(x)\frac{(x-x)^2}{2}}_{0} + f''(0)\frac{x^2}{2} + \int_0^x \frac{(x-t)^2}{2} f^{(3)}(t)\,dt$$

~~Integrate by parts again:~~

$$f(x) = \boxed{f(0) + f'(0)x + f''(0)\frac{x^2}{2} + \int_0^x \frac{(x-t)^2}{2!} f^{(3)}(t)\,dt}$$

$$\int_0^x \underbrace{f^{(3)}(t)}_{u}\,\underbrace{\frac{(x-t)^2}{2!}\,dt}_{dv} = uv\Big|_0^x - \int_0^x v\,du$$

use IBP again...

$$= \underbrace{-f^{(3)}(t)}_{u}\,\underbrace{\frac{(x-t)^3}{3!}}_{-v}\Bigg|_{0=t}^{x=t} + \int_0^x \underbrace{\frac{(x-t)^3}{3!}}_{-v}\,\underbrace{f^{(4)}(t)\,dt}_{du}$$

because $v=\int \frac{(x-t)^2}{2!}\,dt = \int \frac{w^2}{2!}(-dw) = -\frac{w^3}{2!\cdot 3} + c$ ← choose $c=0$

$= -\frac{(x-t)^3}{3!}$, so $-v = (x-t)^3/3!$

Therefore:

$$\boxed{f(x) = f(0) + f'(0)x + f''(0)\frac{x^2}{2!} + \underbrace{f^{(3)}(0)\frac{x^3}{3!}} + \int_0^x f^{(4)}(t)\frac{(x-t)^3}{3!}\,dt}$$

$$\left(-f^{(3)}(x)\overbrace{\frac{(x-x)^3}{3!}}^{0}\right) - \left(-f^{(3)}(0)\frac{(x-0)^3}{3!}\right)$$

Using integration by parts again,

$$\boxed{\begin{aligned} f(x) &= f(0) + f'(0)x + f''(0)\frac{x^2}{2!} \\ &+ f^{(3)}(0)\frac{x^3}{3!} + f^{(4)}(0)\frac{x^4}{4!} + \int_0^x f^{(5)}(t)\frac{(x-t)^4}{4!}\,dt \end{aligned}}$$

The pattern continues:

$$f(x) = \underbrace{\sum_{k=0}^{n}\left(f^{(k)}(0)\frac{x^k}{k!}\right)}_{\text{nth degree Taylor expansion at } x=0} + \underbrace{\int_0^x f^{(n+1)}(t)\frac{(x-t)^n}{n!}\,dt}_{\text{remainder } R_n}$$