13 ① $\langle x,y,z\rangle = \langle 0,0,1\rangle + t\langle 7,6,-1\rangle,$

$$0 \le t \le 1.$$

② $\langle x,y,z\rangle = \langle -1,2,-2\rangle$

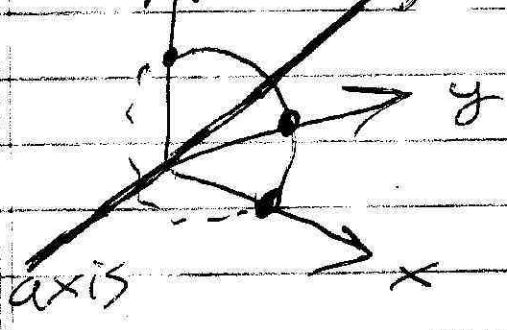
$$+ \frac{4}{\sqrt{14}}\langle 1,2,3\rangle \cos t$$

$$+ \frac{4}{\sqrt{14}}\langle 0,3,-2\rangle \sin t,$$

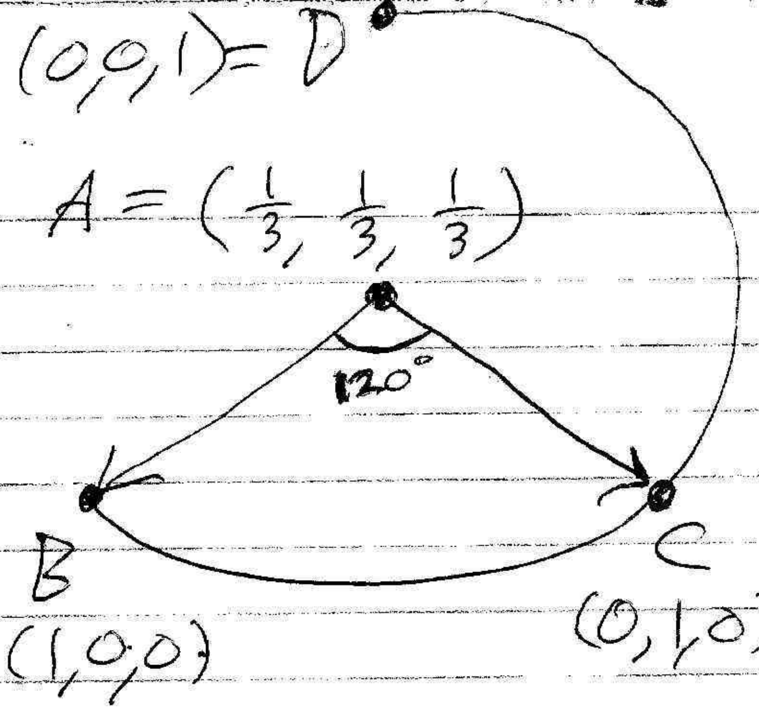
$$0 \le t \le 2\pi.$$

③ $\overbrace{240°}^{4\pi/3}$ circular arc through $(1,0,0)$, $(0,1,0)$ & $(0,0,1)$.

By symmetry, the axis of the circle must be $x=y=z$ (the line). The three given points are on the plane $x+y+z=1$, which intersects the axis at $(1/3, 1/3, 1/3)$, which must be the circle's center.



13



$\overrightarrow{AB} = \left\langle \frac{2}{3}, -\frac{1}{3}, -\frac{1}{3} \right\rangle$

$\overrightarrow{AC} = \left\langle -\frac{1}{3}, \frac{2}{3}, -\frac{1}{3} \right\rangle$

$|\overrightarrow{AB}| = \frac{\sqrt{6}}{3}$

$\vec{u} = \langle 2, -1, -1 \rangle$
$\vec{v} = \langle -1, 2, -1 \rangle$

The circle has radius $\sqrt{6}/3$.
We need two perpendicular unit vectors parallel to the plane of the circle in order to parametrize the circular arc.

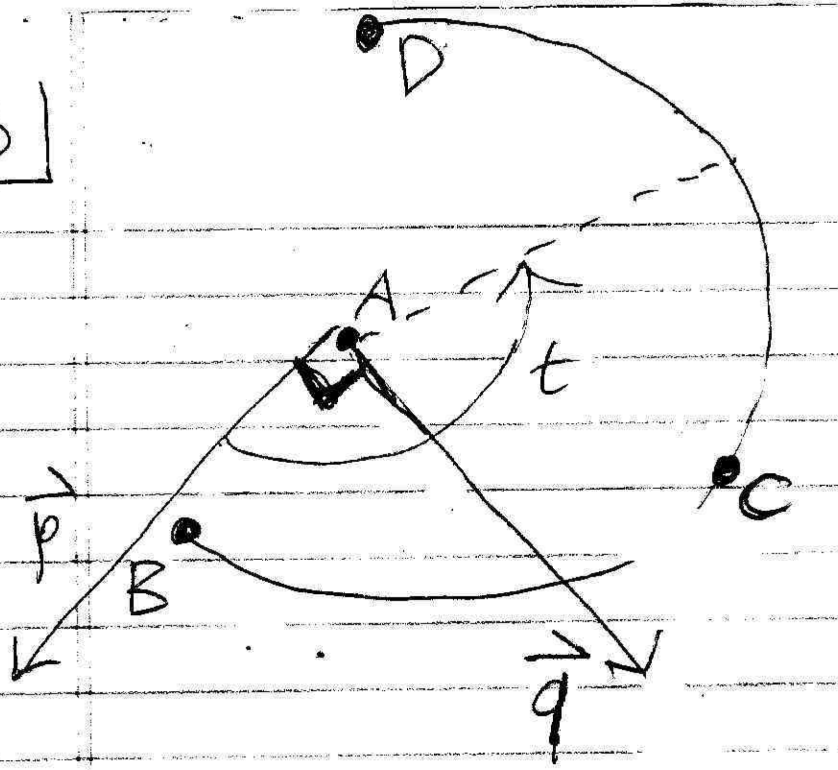
$\vec{u} = 3(\overrightarrow{AB})$ & $\vec{v} = 3(\overrightarrow{AC})$ are parallel to that plane, but are not $\perp$. But the part $\vec{w}$ of $\vec{v}$ perpendicular to $\vec{u}$ is still parallel to the plane.

$$\vec{w} = \vec{v} - \text{proj}_{\vec{u}} \vec{v} = \vec{v} - \frac{\vec{u} \cdot \vec{v}}{|\vec{u}|^2} \vec{u} = \left\langle 0, \frac{3}{2}, -\frac{3}{2} \right\rangle.$$

So, $\vec{u}$ & $\vec{w}$ meet all our requirements except that they are not unit vectors.

Let
$$\begin{cases} \vec{p} = \dfrac{\vec{u}}{|\vec{u}|} = \dfrac{1}{\sqrt{6}} \langle 2, -1, -1 \rangle \\[2ex] \vec{q} = \dfrac{\vec{w}}{|\vec{w}|} = \dfrac{1}{\sqrt{2}} \langle 0, 1, -1 \rangle \end{cases}$$

13



$$R = \frac{\sqrt{6}}{3}$$

$$0 \le t \le \frac{4\pi}{3}$$

$$\vec{p} = \frac{1}{\sqrt{6}}\langle 2, -1, -1\rangle$$

$$\vec{q} = \frac{1}{\sqrt{2}}\langle 0, 1, -1\rangle$$

Finally, since $\vec{p} = \frac{3}{|\vec{u}|}\overrightarrow{AB}$ is parallel to $\overrightarrow{AB}$,

we can parametrize the arc starting at $\vec{B}$ with $t=0$ there as follows:

$$\langle x, y, z\rangle = \left\langle \frac{1}{3}, \frac{1}{3}, \frac{1}{3}\right\rangle + R(\vec{p}\cos t + \vec{q}\sin t)$$

$$= \left\langle \frac{1}{3}, \frac{1}{3}, \frac{1}{3}\right\rangle + \frac{1}{3}\langle 2, -1, -1\rangle \cos(t)$$

$$+ \frac{1}{\sqrt{3}}\langle 0, 1, -1\rangle \sin(t)$$

for $0 \le t \le 4\pi/3$.

IF you prefer non-vector notation:

$$\begin{cases} x = (1 + 2\cos t)/3 \\ y = (1 - \cos t + \sqrt{3}\sin t)/3 \\ z = (1 - \cos t - \sqrt{3}\sin t)/3 \end{cases} \quad \text{for } 0 \le t \le \frac{4\pi}{3}.$$

14 $\vec{r} = \langle (\sin t)(\cos 20t), (\sin t)(\sin 20t), \cos t \rangle$

$\vec{r}\,' = \langle (\cos t)(\cos 20t) - (\sin t)(\sin 20t)(20),$
$(\cos t)(\sin 20t) + (\sin t)(\cos 20t)(20),$
$-\sin t \rangle$

$\vec{T} = \vec{r}\,' / |\vec{r}\,'|$

(1)

t	$\vec{r}\,'$ (approx.)	$\vec{T}$ (approx.)
1	$\langle -15.1, +7.36, -.841 \rangle$	$\langle -.898, +.437, -.0499 \rangle$
2	$\langle -13.3, -12.4, -.909 \rangle$	$\langle -.729, -.683, -.0499 \rangle$
3	$\langle +1.80, -2.39, -.141 \rangle$	$\langle +.602, -.797, -.0471 \rangle$

(2) At $t=1$, $\vec{r} \approx \langle +.343, +.768, +.540 \rangle$.
The requested plane is (approx.):

$$\{(x,y,z) \mid 0 = -.898(x-.343) + .437(y-.768) - .0499(z-.540)\}$$

(3) $$\int_0^\pi |\vec{r}\,'(t)|\,dt = \int_0^\pi \sqrt{(\ldots)^2 + (\ldots)^2 + (\ldots)^2}\,dt$$

≈ 40.2

Note: $|\vec{r}\,'(t)|$ can be simplified to $\sqrt{1+(20\sin t)^2}$, but there is no formula for the integral.

16 ① $\vec{r} = \langle 7\cos t, 4\sin t \rangle = \left\langle \frac{7}{\sqrt{2}}, \frac{4}{\sqrt{2}} \right\rangle$ at $t = \frac{\pi}{4}$

$\vec{r}\,' = \vec{v} = \langle -7\sin t, 4\cos t \rangle = \left\langle -\frac{7}{\sqrt{2}}, \frac{4}{\sqrt{2}} \right\rangle$ at $t = \frac{\pi}{4}$

$|\vec{r}\,'| = |\vec{v}| = \sqrt{49\sin^2 t + 16\cos^2 t} = \sqrt{16 + 33\sin^2 t}$

$$\vec{T} = \frac{\langle -7\sin t, 4\cos t \rangle}{\sqrt{16 + 33\sin^2 t}} = \frac{\langle -7, 4 \rangle}{\sqrt{65}} \text{ @ } \frac{\pi}{4}$$

$$\vec{T}\,' = \left\langle \frac{-7\cos(t)}{|\vec{v}|} + \frac{231\sin^2(t)\cos(t)}{|\vec{v}|^3}, \frac{-4\sin(t)}{|\vec{v}|} - \frac{132\cos^2(t)\sin(t)}{|\vec{v}|^3} \right\rangle$$

$$\left(\vec{T}\,' \text{ at } t = \frac{\pi}{4}\right) = \langle -224, -392 \rangle / 65^{3/2}$$

$$\vec{N} = \vec{T}\,' / |\vec{T}\,'| = \frac{\langle -4, -7 \rangle}{\sqrt{65}} \text{ @ } t = \pi/4$$

$$R = |\vec{r}\,'| / |\vec{T}\,'| = \underbrace{65^{3/2}\sqrt{2} / 112}_{\approx 6.617} \text{ @ } \pi/4$$

Osculating circle: for $0 \le t \le 2\pi$,

$$\langle x, y \rangle = \left\langle \frac{7}{\sqrt{2}}, \frac{4}{\sqrt{2}} \right\rangle + \frac{65\sqrt{2}}{112}\Big(\langle -7, 4 \rangle \cos(t) + \langle -4, -7 \rangle (1 + \sin(t))\Big)$$

16 I used $\vec{r}_{circle} = \vec{r}_{curve} + R(\vec{T}\cos t + \vec{N}(1+\sin(t)))$.

If you prefer non-vector notation:

$$\begin{cases} x = 7/\sqrt{2} + (65\sqrt{2}/112)(-7\cos(t) - 4 - 4\sin(t)) \\ y = 4/\sqrt{2} + (65\sqrt{2}/112)(4\cos(t) - 7 - 7\sin(t)) \\ 0 \le t \le 2\pi \end{cases}$$

[Alternate solution method for 1 & 2: Use the Day 17 techniques involving $|\vec{a}_\perp|$ to find $\vec{N}$ & R.]

2 $\vec{r} = \langle 7\cos t, 4\sin t, t\rangle = \left\langle \frac{7}{\sqrt{2}}, \frac{4}{\sqrt{2}}, \frac{\pi}{4}\right\rangle$ @ $\frac{\pi}{4}$

$\vec{v} = \langle -7\sin t, 4\cos t, 1\rangle = \left\langle -\frac{7}{\sqrt{2}}, \frac{4}{\sqrt{2}}, 1\right\rangle$ @ $\pi/4$

$|\vec{v}| = \sqrt{17 + 33\sin^2 t}$

$|\vec{v}| = \sqrt{67/2}$ @ $\pi/4$

$$\vec{T} = \frac{\langle -7\sin t, 4\cos t, 1\rangle}{\sqrt{17+33\sin^2 t}} = \frac{\langle -7, 4, \sqrt{2}\rangle}{\sqrt{67}} \text{ @ } \pi/4$$

$$\vec{T}' = \left\langle \frac{-7\cos(t)}{|\vec{v}|} + \frac{231\sin^2(t)\cos(t)}{|\vec{v}|^3}, \frac{-4\sin(t)}{|\vec{v}|} - \frac{132\cos^2(t)\sin(t)}{|\vec{v}|^3}, \frac{-33\sin(t)\cos(t)}{|\vec{v}|^3} \right\rangle$$

16

$$\vec{T}' = \langle -238, -400, -33\sqrt{2} \rangle / 67^{3/2}$$

@ $\pi/4$

$$\vec{N} = \vec{T}' / |\vec{T}'| = \frac{\langle -119\sqrt{2}, -200\sqrt{2}, -33 \rangle}{\sqrt{109411}}$$

$$R = |\vec{v}| / |\vec{T}'| = \underbrace{67^{3/2} / (2\sqrt{1633})}_{\approx 6.786} \text{ @ } \frac{\pi}{4}$$

Again using $\vec{r}_{circle} = \vec{r}_{curve} + R(\vec{T}\cos(t) + \vec{N}(1 + \sin(t)))$:

$$\langle x, y, z \rangle = \left\langle \frac{7}{\sqrt{2}}, \frac{4}{\sqrt{2}}, \frac{\pi}{4} \right\rangle$$

$$+ \frac{67}{2\sqrt{1633}} \left(\langle -7, 4, \sqrt{2} \rangle \cos(t) + \langle -119\sqrt{2}, -200\sqrt{2}, -33 \rangle \frac{1 + \sin(t)}{\sqrt{109411}} \right)$$

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16 Approximate solutions:

① $\begin{cases} x \approx 1.67 - 5.75\cos(t) - 3.28\sin(t) \\ y \approx -2.92 + 3.28\cos(t) - 5.75\sin(t) \end{cases}$

② $\begin{cases} x \approx 1.50 - 5.80\cos(t) - 3.45\sin(t) \\ y \approx -2.97 + 3.32\cos(t) - 5.80\sin(t) \\ z \approx 0.108 + 1.17\cos(t) - 0.677\sin(t) \end{cases}$

16 osculating circle centers:

① $(x,y) \approx (1.67, -2.92)$

② $(x,y,z) \approx (1.50, -2.97, .108)$

---

17 $\vec{r} = \langle t, t^2, t^3 \rangle$. $\vec{v} = \langle 1, 2t, 3t^2 \rangle$.

$\vec{a} = \langle 0, 2, 6t \rangle$.

At $t = \frac{1}{3}$, $\vec{v} = \langle \frac{3}{3}, \frac{2}{3}, \frac{1}{3} \rangle$ & $\vec{a} = \langle 0, 2, 2 \rangle$.

$\vec{v} \cdot \vec{a} = 2$ & $|\vec{v}| = \sqrt{14}/3$.

$$\frac{d|\vec{v}|}{dt} = \frac{\vec{v} \cdot \vec{a}}{|\vec{v}|} = \frac{6}{\sqrt{14}}$$

$$\vec{T} = \frac{\vec{v}}{|\vec{v}|} = \frac{\langle 3, 2, 1 \rangle}{\sqrt{14}}$$

$$\vec{a}_{\parallel} = \text{proj}_{\vec{v}} \vec{a} = \frac{\vec{v} \cdot \vec{a}}{|\vec{v}|^2} \vec{v} = \frac{9}{7} \vec{v} = \left\langle \frac{9}{7}, \frac{6}{7}, \frac{3}{7} \right\rangle$$

$$\vec{a}_{\perp} = \vec{a} - \vec{a}_{\parallel} = \left\langle -\frac{9}{7}, \frac{8}{7}, \frac{11}{7} \right\rangle$$

17) $|\vec{a}_\perp| = \sqrt{266}/7$; $R = \dfrac{|\vec{v}|^2}{|\vec{a}_\perp|} = \dfrac{98}{9\sqrt{266}} \approx .67$

$$\vec{N} = \frac{\vec{a}_\perp}{|\vec{a}_\perp|} = \frac{\langle -9, 8, 11\rangle}{\sqrt{266}}$$

...

Comments: In the Zip archive for this solution set I've included Python code & plots relevant to $\underbrace{\text{HW16}}_{\text{code+plots}}$ & $\underbrace{\text{HW14}}_{\text{code}}$.

For HW13, you can check that the curve really goes through the correct points:

			x	y,	z
t	$\cos t$	$\sin t$	$(1+2\cos(t))/3$	$\frac{1}{3}-\frac{1}{3}\cos(t) \pm \frac{1}{\sqrt{3}}\sin(t)$	
0	1	0	1	0	0
$2\pi/3$	$-1/2$	$\sqrt{3}/2$	0	1	0
$4\pi/3$	$-1/2$	$-\sqrt{3}/2$	0	0	1