HW25

Given: $\frac{d\rho}{dt}=0$; $\frac{dx}{dt}=5$; $\frac{dy}{dt}=-7$.

$\rho=\sqrt{x^2+y^2+z^2}$; currently $(\rho,\varphi,\theta)=(5,1,4)$

$$\frac{\partial\rho}{\partial x}=\frac{x}{\sqrt{x^2+y^2+z^2}}=\frac{x}{\rho};\ \frac{\partial\rho}{\partial y}=\frac{y}{\rho};\ \frac{\partial\rho}{\partial z}=\frac{z}{\rho}$$

$$\left[\begin{aligned}0=\frac{d\rho}{dt}&=\frac{\partial\rho}{\partial x}\frac{dx}{dt}+\frac{\partial\rho}{\partial y}\frac{dy}{dt}+\frac{\partial\rho}{\partial z}\frac{dz}{dt}\\&=\frac{x}{\rho}\cdot5+\frac{y}{\rho}(-7)+\frac{z}{\rho}\frac{dz}{dt}\end{aligned}\right.$$

$\frac{x}{\rho}=\sin\varphi\cos\theta=\sin1\cos4$ (currently)

$\frac{y}{\rho}=\sin\varphi\sin\theta=\sin1\sin4$ (currently)

$\frac{z}{\rho}=\cos\varphi=\cos1$ (currently)

Solving for current value of dz/dt:

$$\frac{dz}{dt}=(\tan1)(7\sin4-5\cos4)$$
$$\approx-3.1606$$

$z=\rho\cos\varphi$ $\qquad$ $0=d\rho/dt$

$$\frac{dz}{dt}\overset{\Downarrow}{=}\frac{\partial z}{\partial\rho}\frac{d\rho}{dt}+\frac{\partial z}{\partial\varphi}\frac{d\varphi}{dt}\overset{\Downarrow}{=}0-\rho(\sin\varphi)\frac{d\varphi}{dt}$$

$$\Rightarrow\frac{d\varphi}{dt}=\frac{-dz/dt}{\rho\sin\varphi}=\frac{-(\tan1)(7\sin4-5\cos4)}{5\sin1}$$

≈0.7512 (currently)

Last, we find $d\theta/dt$ using $y=\rho\sin\varphi\sin\theta$.

currently

$$-7 \overset{\downarrow}{=} \frac{dy}{dt} = \frac{\partial y}{\partial \rho}\frac{d\rho}{dt} + \frac{\partial y}{\partial \varphi}\frac{d\varphi}{dt} + \frac{\partial y}{\partial \theta}\frac{d\theta}{dt}$$

$$= 0 + (\rho \cos\varphi \sin\theta)\frac{d\varphi}{dt} + (\rho \sin\varphi \cos\theta)\frac{d\theta}{dt}$$

$$\Uparrow \qquad \uparrow \quad \uparrow \quad \uparrow \quad \uparrow \quad \uparrow \quad \uparrow \quad \uparrow$$

currently: $0 = d\rho/dt \quad 5 \quad 1 \quad 4 \quad .7512\ldots \quad 5 \quad 1 \quad 4$

Solving for $d\theta/dt$:

$$-7 = (5\cos 4 - 7\sin 4)(\sin 4) + 5(\sin 1)(\cos 4)(d\theta/dt)$$

$$\Rightarrow \frac{d\theta}{dt} = \frac{(7\sin 4)(7\sin 4 - 5\cos 4)}{(5\cos 4)(\sin 1)}$$

$$\approx 1.196 \text{ (currently)}$$

HW26 $f(x,y) = x^3y^2 + x - y$, $(a,b) = (1,4)$, & $\vec{u} = \langle \cos\theta, \sin\theta \rangle$ are given.

① $(D_{\vec{u}} f)(a,b) = \vec{\nabla} f \cdot \vec{u}$ @ (a,b)

$$= \langle \underbrace{3x^2y^2 + 1}_{\partial f/\partial x}, \underbrace{2x^3y - 1}_{\partial f/\partial y} \rangle \cdot \langle \cos\theta, \sin\theta \rangle \quad @ (x,y) = (a,b)$$

$$= \langle 49, 7 \rangle \cdot \langle \cos\theta, \sin\theta \rangle$$
$$= \boxed{49\cos\theta + 7\sin\theta}$$

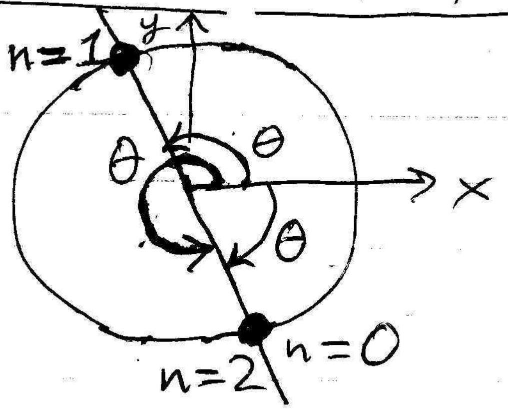
② $0 = (D_{\vec{u}} f)(a,b) = 49\cos\theta + 7\sin\theta$

$$\Longleftrightarrow -49\cos\theta = 7\sin\theta \Longleftrightarrow -7 = \frac{\sin\theta}{\cos\theta}$$
$$= \tan\theta \Longleftrightarrow \theta = \tan^{-1}(-7) + n\pi$$
$$= \boxed{-\tan^{-1}(7) + n\pi \quad (\text{for } n = 0, \pm 1, \pm 2, \ldots)}$$

A picture may help:

$\tan\theta = \text{slope} = -7$;

$n = 0, 1, 2$ shown



③ $\cos\theta = \dfrac{1}{\sec\theta} = \dfrac{\pm 1}{\sqrt{1+\tan^2\theta}} = \dfrac{\pm 1}{\sqrt{50}}$

$\sin\theta = \dfrac{\tan\theta}{\sec\theta} = \dfrac{\pm\tan\theta}{\sqrt{1+\tan^2\theta}} = \dfrac{\mp 7}{\sqrt{50}}$

So, $\vec{u} = \pm \dfrac{1}{\sqrt{50}} \langle 1, -7 \rangle$.

HW26 Comment: What's the point of ② & ③? Where $D_{\vec{u}} f = 0$, we are parallel to the tangent line of the level curve. So, ② & ③ find the two unit vectors parallel to the tangent line @ (a,b), first in terms of polar coordinates, and then in x & y components.

$f(x,y) = x\sin(x+y)$

HW27 $\vec{\nabla} f = \langle f_x, f_y \rangle = \langle \sin(x+y) + x\cos(x+y),$
$x\cos(x+y) \rangle$

Tangent line at $(1,2)$:

$0 = \vec{\nabla} f(1,2) \cdot \langle x-1, y-2 \rangle$

$\iff 0 = (\sin 3 + \cos 3)(x-1) + (\cos 3)(y-2)$

@ $(1,2)$, max. slope $= |\vec{\nabla} f(1,2)| = \sqrt{(\sin 3 + \cos 3)^2 + \cos^2 3}$
≈ 1.304

direction of max. slope @ $(1,2)$

$= \dfrac{\vec{\nabla} f(1,2)}{|\vec{\nabla} f(1,2)|} = \left\langle \dfrac{\sin 3 + \cos 3}{\sqrt{\cdots}}, \dfrac{\cos 3}{\sqrt{\cdots}} \right\rangle$

$\approx \langle -.6509, -.7591 \rangle$

$g(x,y,z) = (z-y)(y-x)^{-1}$

$\vec{\nabla} g = \langle g_x, g_y, g_z \rangle = \langle (z-y)(y-x)^{-2},$
$(x-z)(y-x)^{-2},$
$(y-x)^{-1} \rangle$

$\vec{\nabla} g(1,2,3) = \langle 1, -2, 1 \rangle$

Tangent plane @ $(1,2,3)$:

$0 = \vec{\nabla} g(1,2,3) \cdot \langle x-1, y-2, z-3 \rangle$

$\iff 0 = (x-1) - 2(y-2) + (z-3)$

min. directional deriv. @ $(1,2,3)$
$= -|\vec{\nabla} g(1,2,3)| = -\sqrt{6}$. The direction for this minimum is $-\vec{\nabla} g / |\vec{\nabla} g|$ @ $(1,2,3)$, which equals $\langle -1/\sqrt{6}, +2/\sqrt{6}, -1/\sqrt{6} \rangle$.

HW28 ① ↓

Given: $z = x^3 - 10x + y^3 - 10y + 5$

$z_x = 3x^2 - 10 + 0 - 0 + 0$

$z_y = 0 - 0 + 3y^2 - 10 + 0$

4 Critical points: $\pm\left(\sqrt{\frac{10}{3}}, \sqrt{\frac{10}{3}}\right)$ & $\pm\left(\sqrt{\frac{10}{3}}, -\sqrt{\frac{10}{3}}\right)$

$z_{xx} = 6x$; $D = z_{xx} z_{yy} - z_{xy}^2 = (6x)(6y) - 0^2$

Crit. pt.	$D = 36xy$	$z_{xx} = 6x$	classification
$(+\sqrt{10/3}, +\sqrt{10/3})$	+	+	@ local min.
$(+\sqrt{10/3}, -\sqrt{10/3})$	−		saddle pt.
$(-\sqrt{10/3}, +\sqrt{10/3})$	−		saddle pt.
$(-\sqrt{10/3}, -\sqrt{10/3})$	+	−	@ local max.

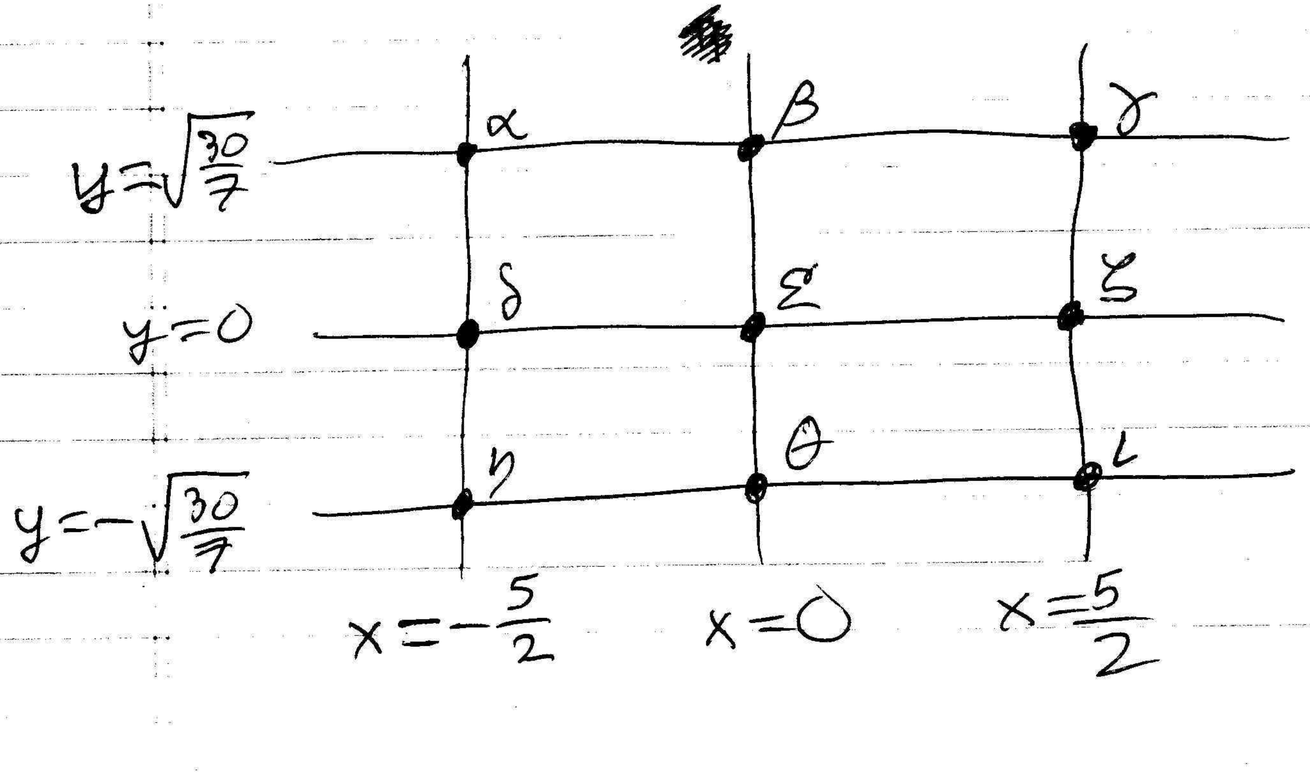
②

$z = f(x,y) = 8x^4 - 100x^2 + 7y^4 - 60y^2 + 5$

$f_x = 32x^3 - 200x = 8x \cdot (4x^2 - 25)$

$f_y = 28y^3 - 120y = 4y \cdot (7y^2 - 30)$

Solve $f_x = f_y = 0$:



$f_{xx} = 96x^2 - 200; \quad f_{xy} = 0$

$f_{yy} = 84y^2 - 120$

$D = (96x^2 - 200)(84y^2 - 120) - 0^2$

	(x,y) = Point	D	f_{xx}	classification	$f(x,y)$
alpha	α	+	+	@ loc. min.	$5 - 625/2 - 900/7$
beta	β	−		saddle pt.	
gamma	γ	+	+	@ loc. min.	$5 - 625/2 - 900/7$
delta	δ	−		saddle pt.	
epsilon	ε	+	−	@ loc. max	5
zeta	ζ	−		saddle pt.	
eta	η	+	+	@ loc. min	$5 - 625/2 - 900/7$
theta	θ	−		saddle pt.	
iota	ι	+	+	@ loc. min	$5 - 625/2 - 900/7$

$f = 4x^2(2x^2 - 25)$
$+ y^2(7y^2 - 60)$
$+5$

The local max. is 5

The local min. is $5 - \frac{625}{2} - \frac{900}{7}$