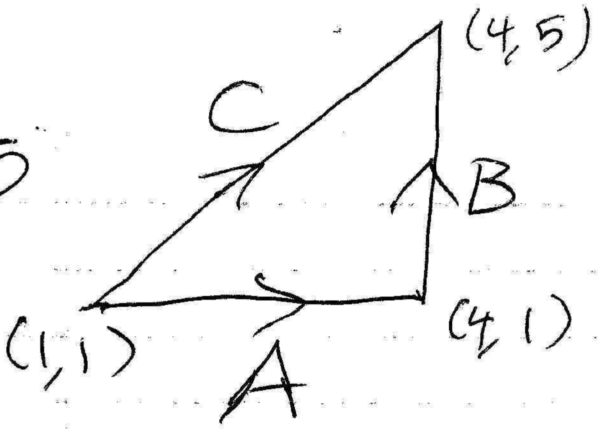
HW45



- $y=1$ & $dy=0$ on A.
- $x=4$ & $dx=0$ on B.
- Parametrizing C:

$$\begin{cases} x = 1+(4-1)t \\ y = 1+(5-1)t \\ 0<t<1 \end{cases}$$

$d\vec{r} = \langle dx, dy \rangle$ $\qquad$ $dx = 3dt$ & $dy = 4dt$ on C.

$$\vec{F} = \langle x, y \rangle; \int_{A+B} \vec{F}\cdot d\vec{r} = \int_{x=1}^{x=4} x\,dx + \int_{y=1}^{y=5} y\,dy$$

$$= \frac{15}{2} + \frac{24}{2} = \boxed{\frac{39}{2}}; \int_C \vec{F}\cdot d\vec{r} = \int_0^1 \Big[\underbrace{(1+3t)}_{x}\underbrace{3}_{dx/dt} + \underbrace{(1+4t)}_{y}\underbrace{4}_{dy/dt}\Big]dt$$

$$= 7 + \frac{25}{2} = \boxed{\frac{39}{2}}.$$

$$\vec{G} = \langle y, x \rangle; \int_{A+B} \vec{G}\cdot d\vec{r} = \int_1^4 \underbrace{1}_{y}\,dx + \int_1^5 \underbrace{4}_{x}\,dy = \boxed{19};$$

$$\int_C \vec{G}\cdot d\vec{r} = \int_0^1 \Big[\underbrace{(1+4t)}_{y}\underbrace{3}_{dx/dt} + \underbrace{(1+3t)}_{x}\underbrace{4}_{\frac{dy}{dt}}\Big]dt = 7+12 = \boxed{19}.$$

$$\vec{H} = \langle -y, x \rangle; \int_{A+B} \vec{H}\cdot d\vec{r} = -\int_1^4 1\,dx + \int_1^5 4\,dy = \boxed{13};$$

$$\int_C \vec{H}\cdot d\vec{r} = \int_0^1 \Big[\underbrace{-(1+4t)}_{-y}3 + \underbrace{(1+3t)}_{x}4\Big]dt = \boxed{1}.$$

$$\vec{J} = \langle x^2-y, x+y^2 \rangle; \int_{A+B} \vec{J}\cdot d\vec{r} = \int_1^4 (x^2-1)\,dx + \int_1^5 (4+y^2)\,dy$$

$$= 63/3 - 3 + 16 + 124/3 = 21 + 13 + 41 + \tfrac{1}{3} = \boxed{75 + \tfrac{1}{3}};$$

$$\int_C \vec{J}\cdot d\vec{r} = \int_0^1 \Big[\big(\underbrace{(1+3t)^2 - (1+4t)}_{x^2-y}\big) \underbrace{3}_{dx/dt} + \big(\underbrace{(1+3t)}_{x} + \underbrace{(1+4t)^2}_{y^2}\big) \underbrace{4}_{dy/dt} \Big] dt$$

$$= \boxed{63 + \frac{1}{3}}.$$ $\vec{K} = \langle x^2+y, x+y^2 \rangle$;

$$\int_{A+B} \vec{K}\cdot d\vec{r} = \int_1^4 (x^2+1)\,dx + \int_1^5 (4+y^2)\,dy$$

$= \boxed{81 + \frac{1}{3}}$; $\int_C \vec{K}\cdot d\vec{r} = \int_0^1 \Big[\big((1+3t)^2 + (1+4t)\big) 3$

$+ \big((1+3t) + (1+4t)^2\big) 4 \Big] = \boxed{81 + \frac{1}{3}}$.

HW4 7 ① $\langle P, Q \rangle = \langle x, y \rangle / (x^2+y^2)$.

$$\frac{\partial Q}{\partial x} = \frac{-2xy}{(x^2+y^2)^2} = \frac{\partial P}{\partial y}.$$ To solve $\vec{\nabla} f$

$= \langle f_x, f_y \rangle = \langle P, Q \rangle$: $f = \int P\,dx = \int \frac{x\,dx}{x^2+y^2}$ y held constant

$= \frac{1}{2}\log(x^2+y^2) + g(y)$

$\Rightarrow \frac{y}{x^2+y^2} = Q = f_y = \frac{y}{x^2+y^2} + g'(y)$

$\Rightarrow \boxed{f = \frac{1}{2}\log(x^2+y^2) + c}$

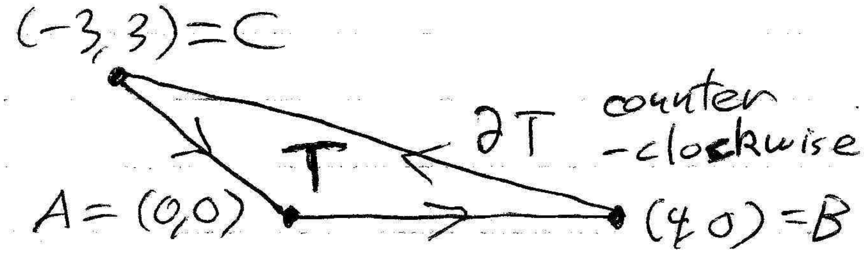
HW47 ② $\langle P, Q\rangle = \langle y, x\rangle / (x^2+y^2)$.

$$\frac{\partial Q}{\partial x} = \frac{y^2-x^2}{(x^2+y^2)^2} \neq \frac{x^2-y^2}{(x^2+y^2)^2} = \frac{\partial P}{\partial y}$$

So, $\langle P, Q\rangle$ is not a gradient.

Let T be the interior of triangle ABC

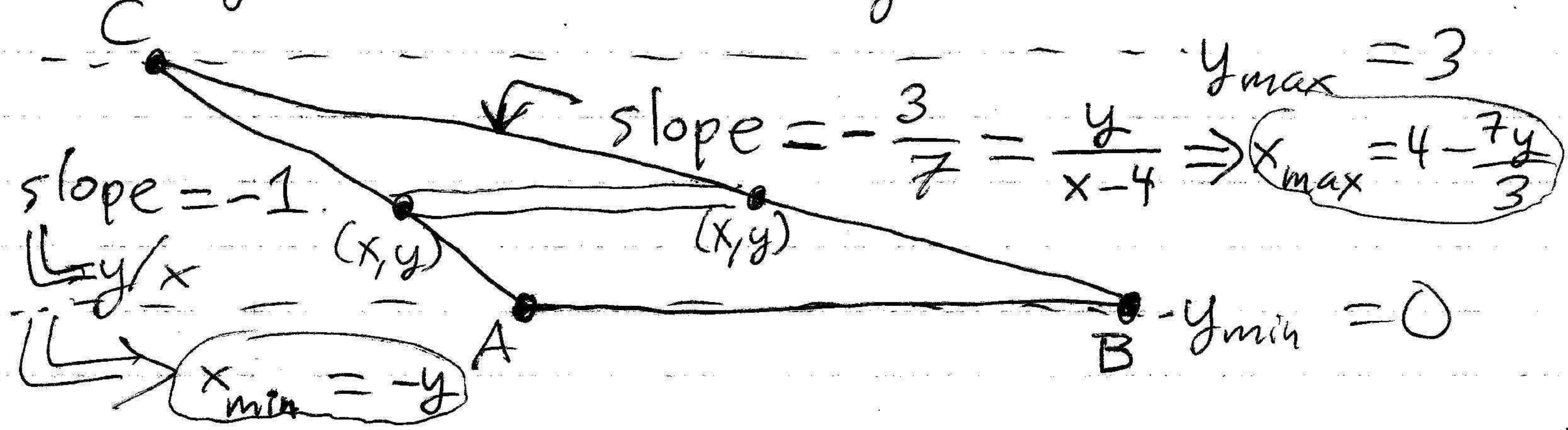
HW48 where $A=(0,0)$, B $4,0)$, $C=(-3,3)$.



$$\int_{\underbrace{ABCA}_{\partial T}} \Big(\underbrace{(x^2+y^2)}_{P}dx + \underbrace{xy}_{Q}\,dy\Big) = \iint_T \underbrace{(y-2y)}_{Q_x - P_y}\,dA$$

$$= \int_{y=0}^{y=3}\left[\int_{x=-y}^{x=4-(7/3)y} (-y)\,dx\right] dy = \boxed{-6}$$

using horizontal slicing:



$$D = \{(x,y) \mid 3 \le r \le 4\}$$

HW50 $\vec{F} = \langle x^2 + y, x - y^2 \rangle = \langle P, Q \rangle$

$$\int_{\partial D} \vec{F} \cdot \underbrace{\vec{T}\, ds}_{\langle dx, dy \rangle} = \iint_D (Q_x - P_y)\, dA$$

$$= \iint_D (1 - 1)\, dA = \boxed{0}.$$

$$\int_{\partial D} \vec{F} \cdot \underbrace{\vec{N}\, ds}_{\langle dy, -dx \rangle} = \iint_D (P_x + Q_y)\, dA$$

$$= \iint_D (2x - 2y)\, dA = \int_3^4 \left[\int_0^{2\pi} 2r(\cos\theta - \sin\theta) \cdot r\, d\theta \right] dr = \int_3^4 2r^2\, dr \cdot \underbrace{\int_0^{2\pi} (\cos\theta - \sin\theta)\, d\theta}_{0}$$

$$= \boxed{0}.$$

HW52 ① $\vec{r} = \langle x, y, x^2 - y^2 \rangle$; $-1 \le x \le 1$; $-1 \le y \le 1$.

surface area $= \iint_{[-1,1]^2} \left| \langle 1, 0, 2x \rangle \times \langle 0, 1, -2y \rangle \right| dx\, dy$

$$= \int_{-1}^{1} \left(\int_{-1}^{1} \sqrt{4x^2 + 4y^2 + 1}\, dx \right) dy$$

$$\approx \boxed{7.446}$$

HW52 2 $\vec{r} = \langle (7+3\cos u)\cos\theta, (7+3\cos u)\sin\theta,$
$3\sin u\rangle;\ 0 \le u \le 2\pi;\ \ 0 \le \theta \le 2\pi.$

$\vec{r}_u \times \vec{r}_\theta = \langle -3\sin u\cos\theta, 3\sin u\sin\theta\rangle, 3\cos u\rangle$
$\times \langle -\sin\theta, \cos\theta, 0\rangle (3\cos u + 7)$ $\boxed{\begin{array}{c}\text{using} \\ 1 = \cos^2 + \sin^2\end{array}}$

$= (3)\underbrace{(3\cos u + 7)}_{\ge 7-3>0}\underbrace{\langle -\cos u\cos\theta, -\cos u\sin\theta, -\sin u\rangle}_{\text{has magnitude } 1}$

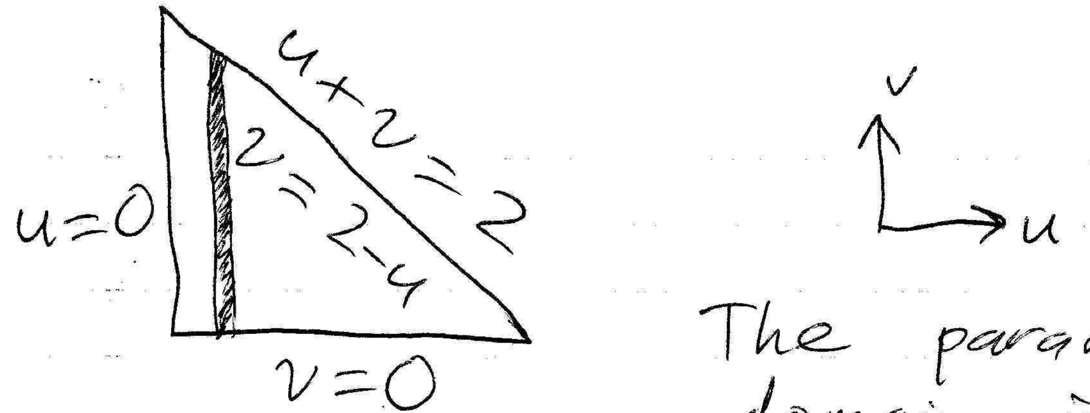
after simplifying $\sqrt{\cdots}$
using $\cos^2 + \sin^2 = 1.$

surface area $= \int_0^{2\pi}\left[\int_0^{2\pi} (7+3\cos u)(3)\,d\theta\right] du$

$$\boxed{= (7)(3)(2\pi)^2 \approx 829}$$

HW52 3 $\vec{r} = \langle u-v, u^2-v^2, u^3-v^3\rangle;$
$0 \le u;\ 0 \le v;\ u+v \le 2.$

$\vec{r}_u \times \vec{r}_v = \langle 1, 2u, 3u^2\rangle \times \langle -1, -2v, -3v^2\rangle$
$= \langle 6uv(u-v), 3(u+v)(v-u), 2(u-v)\rangle.$



The parameters' domain is a triangle.

Vertical slicing $\Rightarrow$ surface area $= \int_0^2 \left(\int_0^{2-u} |\vec{r}_u \times \vec{r}_v| \, dv \right) du$

$$= \int_0^2 \left(\int_0^{2-u} \sqrt{\left[(6uv)^2 + (3(u+v))^2 + 2^2\right](u-v)^2} \, dv \right) du$$

$$\approx \boxed{7.177}$$

Note: HW52 1, 3 don't have exact formulae. You can only estimate them using a calculator or computer.

$\varphi \le \pi/2$

HW53/54 $S = \{(x,y,z) \mid \rho = 4 \ \& \ \overbrace{z \ge 0}\}$

& $\vec{N} = \langle 0, 0, 1 \rangle$ @ $(x,y,z) = (0,0,4)$

& $\vec{F} = \langle z, 0, x^2 \rangle$. $\iint_S \vec{F} \cdot \vec{N} \, dA = ?$

Spherically parametrize S:

$x = 4 \sin\varphi \cos\theta$ $\quad 0 \le \varphi \le \pi/2$

$y = 4 \sin\varphi \sin\theta$ $\quad 0 \le \theta \le 2\pi$

$z = 4 \cos\varphi$

$\vec{r}_\varphi = 4 \langle \cos\varphi \cos\theta, \cos\varphi \sin\theta, -\sin\varphi \rangle$

$\vec{r}_\theta = 4 \langle -\sin\varphi \sin\theta, +\sin\varphi \cos\theta, 0 \rangle$

$\vec{r}_\varphi \times \vec{r}_\theta = 16 \sin\varphi \langle \sin\varphi \cos\theta, \sin\varphi \sin\theta, \cos\varphi \rangle$

using $\cos^2\theta + \sin^2\theta = 1$

$|\vec{r}_\varphi \times \vec{r}_\theta| = 16 \sin\varphi \sqrt{\cdots} = 16 \sin\varphi \cdot 1$

after simplifying.

So, $\vec{N} = \pm \langle \sin\varphi \cos\theta, \sin\varphi \sin\theta, \cos\varphi \rangle$

At $(x,y,z) = (0,0,4)$, we have

$4 = z = 4\cos\varphi \Rightarrow \cos\varphi = 1 \Rightarrow \sin\varphi = 0$

$\Rightarrow \vec{N} = \pm\langle 0,0,1\rangle$. So, "+" gives

$\vec{N} = \langle 0,0,1\rangle$ at $(x,y,z) = (0,0,4)$.

Finally, $\vec{N}\,dA = +16\sin\varphi\langle \sin\varphi\cos\theta,$

$\sin\varphi\sin\theta, \cos\varphi\rangle\, d\varphi\, d\theta$ &

$$\vec{F} = \langle \underbrace{4\cos\varphi}_{z}, 0, (\underbrace{4\cos\theta\sin\varphi}_{x})^2\rangle$$

imply

$$\iint_S \vec{F}\cdot\vec{N}\,dA = \int_0^{\pi/2}\left(\int_0^{2\pi} 16 \cdot \sin\varphi \cdot (4\cos\varphi\sin\varphi\cos\theta + 16\sin^2\varphi\cos\varphi\cos^2\theta)\,d\theta\right)d\varphi$$

$$= \int_0^{\pi/2} (16\sin\varphi)(0 + (16\sin^2\varphi\cos\varphi)\pi)\,d\varphi$$

$$= \int_0^1 16^2\pi \underbrace{u^3}_{\sin^3\varphi} \underbrace{du}_{\cos\varphi\, d\varphi} = \boxed{64\pi}$$

HW55 $S = \{(x,y,z) \mid \overbrace{x^2+y^2+z^2=5^2}^{\rho=5}$

& $\underbrace{z \geq 0}_{\varphi \leq \pi/2}\}$ is parametrized

much like in HW53/54:

$$x = 5 \sin\varphi \cos\theta \qquad 0 \leq \varphi \leq \pi/2$$
$$y = 5 \sin\varphi \sin\theta \qquad 0 \leq \theta \leq 2\pi$$
$$z = 5 \cos\varphi$$

The computation of $\vec{N}\,dA$ is similar too:

$$\vec{N}\,dA = 25 \sin\varphi \langle \sin\varphi\cos\theta, \sin\varphi\sin\theta, \cos\varphi\rangle.$$

This time, $\vec{F} = \langle y, z, x\rangle = 5\langle \sin\varphi\sin\theta, \cos\varphi, \sin\varphi\cos\theta\rangle$.

Direct computation of $\int_{\partial S} \vec{F} \cdot \underbrace{\vec{T}\,ds}_{\langle dx, dy, dz\rangle}$

is accomplished by

parametrizing ∂S using $0 \leq \theta \leq 2\pi$ & $\varphi = \frac{\pi}{2}$:

$\langle x, y, z\rangle = \langle 5\cos\theta, 5\sin\theta, 0\rangle$ &

$\langle dx, dy, dz\rangle = \langle -5\sin\theta, 5\cos\theta, 0\rangle\, d\theta$.

$$\int_{\partial S} \vec{F} \cdot \vec{T}\, ds = \int_0^{2\pi} (-25 \sin^2 \theta)\, d\theta = \boxed{-25\pi}$$

$$\vec{F} = \langle 5 \sin\theta, 0, 5\cos\theta \rangle \text{ at } \varphi = \pi/2.$$

Using Stokes' Theorem, $\int_{\partial S} \vec{F} \cdot \vec{T}\, ds$

$$= \iint_S \underbrace{(\text{curl } \vec{F})}_{\langle -1, -1, -1 \rangle = \text{"}\langle \frac{\partial}{\partial x}, \frac{\partial}{\partial y}, \frac{\partial}{\partial z} \rangle \times \langle y, z, x \rangle\text{"}} \cdot \vec{N}\, dA$$

$$= -\int_0^{2\pi} \left(\int_0^{\pi/2} (25 \sin\varphi)(\sin\varphi(\cos\theta + \sin\theta) + \cos\varphi)\, d\varphi \right) d\theta$$

$$= -25 \Bigg[\int_0^{\pi/2} \sin^2\varphi\, d\varphi \underbrace{\int_0^{2\pi} (\cos\theta + \sin\theta)\, d\theta}_{0}$$

$$+ \underbrace{\int_0^{\pi/2} \cos\varphi \underbrace{\sin\varphi}_{u}\, d\varphi}_{\int_0^1 u\, du = 1/2} \underbrace{\int_0^{2\pi} d\theta}_{2\pi} \Bigg]$$

$$= \boxed{-25\pi}.$$