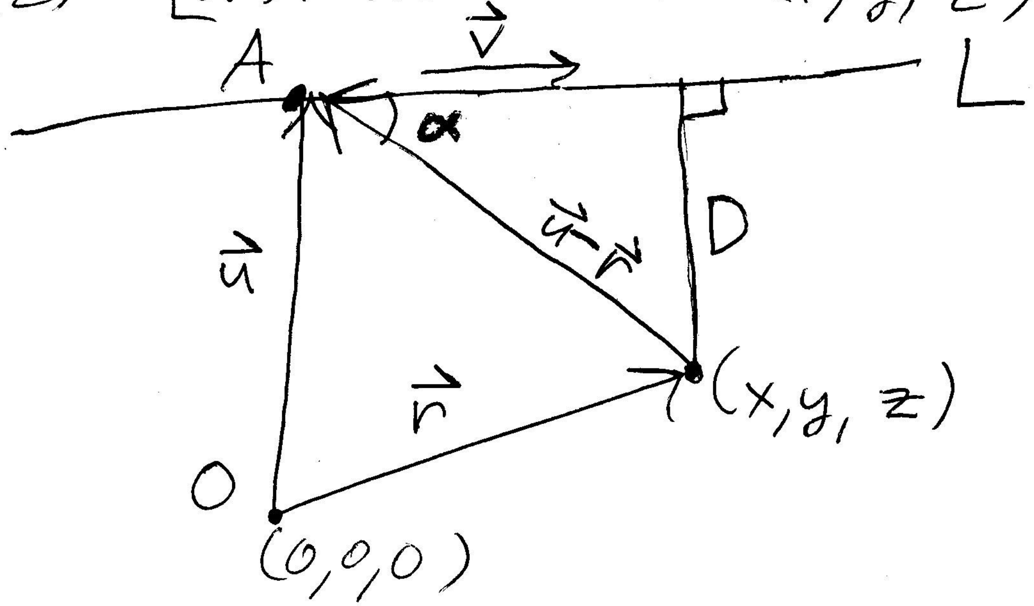
Moment of inertia I with respect to an arbitrary axis L:

$$I = \iiint (\text{distance to } L)^2 \, dm \qquad (\text{3D version})$$

Example: L is the line through $(1,1,1)$ & $(3,4,5)$.

$\mathbf{A} = (1,1,1) \in L \parallel \langle 2,3,4\rangle = \vec{v}$ $\qquad \vec{u} = \overrightarrow{OA} = \langle 1,1,1\rangle$

$D(x,y,z) = [\text{distance from } (x,y,z) \text{ to } L]$



$D = |\vec{u}-\vec{r}| \sin\alpha$

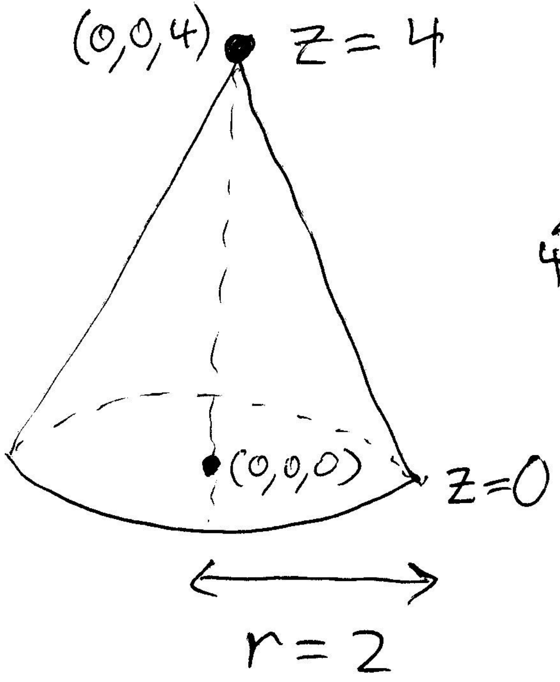
$D = |(\vec{u}-\vec{r}) \times \vec{v}| / |\vec{v}|$

$\vec{u}-\vec{r} = \langle 1-x, 1-y, 1-z\rangle$

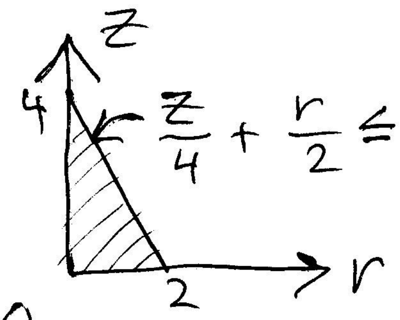
$|\vec{v}| = \sqrt{2^2+3^2+4^2}$

$$D = |\langle (1-y)4-(1-z)3,\ (1-z)2-(1-x)4,\ (1-x)3-(1-y)2 \rangle| / \sqrt{29}$$

$$D^2 = [(3z-4y+1)^2 + (4x-2z-2)^2 + (2y-3x+1)^2]/29$$



$$C = \{(x,y,z) \mid 0 \le z \le 4-2r\}$$



$$\frac{z}{4} + \frac{r}{2} \le 1 \iff z \le 4-2r$$

Given the above cone C, we will find $I = \iiint_C D^2\, dm$, assuming

$$\frac{dm}{dV} = 4 - z.$$

We will use cylindrical coordinates.

$$dV = r\,dr\,d\theta\,dz; \qquad \iiint_C = \int_{\theta=0}^{\theta=2\pi} \int_{r=0}^{r=2} \int_{z=0}^{z=4-2r}$$

$$D^2 = \frac{1}{29}\Big[(3z - 4r\sin\theta + 1)^2 + (4r\cos\theta - 2z - 2)^2$$
$$+ (2r\sin\theta - 3r\cos\theta + 1)^2\Big]$$

$$dm = (4-z)\,dz \cdot \mathbf{r}\,\mathbf{dr}\,d\theta$$

$$I = \int_0^{2\pi}\left[\int_0^{2}\left[\int_0^{4-2r} \frac{1}{29}[\ldots](4-z)\,dz\right] r\,dr\right] d\theta$$

$$I = 14656\pi/435 \approx 105.8$$