HW23

1. If $\vec{r} = \langle t^3 - t, t^5 + t + 1 \rangle$ and $z = x^2 y^3 + 3x + 4y$, then

$$\frac{dz}{dt} = ? \quad \text{at} \quad t = 1 ?$$

2. Assume a particle is moving such that it is always on the surface

$$z = (x^2 + y^4)^{1/2}$$

a) At $(x,y) = (6,1)$, $\frac{dx}{dt} = -3 = \frac{dy}{dt} \Rightarrow \frac{dz}{dt} = ?$

b) At $(x,y) = (-1,2)$, $\frac{dz}{dt} = 5 = \frac{dy}{dt} \Rightarrow \frac{dx}{dt} = ?$

HW2.3

① $\frac{dz}{dt} = \frac{\partial z}{\partial x}\frac{dx}{dt} + \frac{\partial z}{\partial y}\frac{dy}{dt}$

$$\begin{tikzcd} & x \arrow[dr, "z_x"] & \\ t \arrow[ur, "x_t"] \arrow[dr, "y_t"'] & & z \\ & y \arrow[ur, "z_y"'] & \end{tikzcd}$$

$\frac{dz}{dt} = \left(2xy^3+3\right)\left(3t^2-1\right) + \left(3x^2y^2+4\right)\left(5t^4+1\right)$

$= (2(t^3-t)(t^5+t \ldots$ Better to just substitute $t=1$

now: $x = t^3 - t = 0$, $y = t^5+t+1 = 3$, and

$\frac{dz}{dt} = (0+3)(3-1) + (0+4)(5+1) = 6+24 = \boxed{30}$

② $\frac{dz}{dt} = \frac{\partial z}{\partial x}\frac{dx}{dt} + \frac{\partial z}{\partial y}\frac{dy}{dt} = \frac{x}{\sqrt{x^2+y^4}}\frac{dx}{dt} + \frac{2y^3}{\sqrt{x^2+y^4}}\frac{dy}{dt}$

a) $\frac{dz}{dt} = \frac{1}{\sqrt{36+1}}\left(\underset{\substack{\uparrow \\ x}}{6} \cdot \underset{\substack{\uparrow \\ x_t}}{-3} + \underset{\substack{\uparrow \\ 2y^3}}{2} \cdot \underset{\substack{\uparrow \\ y_t}}{-3}\right) = \boxed{\frac{-24}{\sqrt{37}}} \approx -3.95$

b) $5\sqrt{17} = \underset{\substack{\uparrow \\ 5}}{\frac{dz}{dt}} \sqrt{\underset{\substack{\uparrow \\ -1}}{x}^2 + \underset{\substack{\uparrow \\ 2}}{y}^4} = x x_t + 2\underset{\substack{\uparrow \\ 2}}{y}^3 \underset{\substack{\uparrow \\ 5}}{y_t} = -x_t + 80$

$\Rightarrow dx/dt = x_t = \boxed{80 - 5\sqrt{17}} \approx 59.4$

HW24

① If $x = \sin\varphi\cos\theta$ & $y = \sin\varphi\sin\theta$,
and, at $(\varphi, \theta) = (\pi/6, \pi/3)$, $z = f(x,y) = g(\varphi, \theta)$
satisfies $\frac{\partial z}{\partial x} = 3$, $\frac{\partial z}{\partial y} = 4$, then $\frac{\partial z}{\partial \varphi}, \frac{\partial z}{\partial \theta} = ?, ?$

② If instead $\frac{\partial z}{\partial \varphi} = 3$, $\frac{\partial z}{\partial \theta} = 4$, then $\frac{\partial z}{\partial x}, \frac{\partial z}{\partial y} = ?, ?$
(Still at $(\varphi, \theta) = (\pi/6, \pi/3)$.)

③ If $z = f(x,y)$, $x = t\cos(3t)$, $y = g(t)$,
and, at $t = \frac{\pi}{12}$, $\frac{\partial z}{\partial y} = -1$ and $\frac{dy}{dt} = 7 = \frac{dz}{dt}$,
then $\frac{\partial z}{\partial x} = ?$

HW 24

(1)

$$\begin{cases} z_\varphi = \dfrac{\partial z}{\partial \varphi} = \dfrac{\partial z}{\partial x}\dfrac{\partial x}{\partial \varphi} + \dfrac{\partial z}{\partial y}\dfrac{\partial y}{\partial \varphi} = 3\cos\underset{\pi/6}{\varphi}\cos\underset{\pi/3}{\theta} + 4\cos\underset{\pi/6}{\varphi}\sin\underset{\pi/3}{\theta} \\ z_\theta = \dfrac{\partial z}{\partial \theta} = \dfrac{\partial z}{\partial x}\dfrac{\partial x}{\partial \theta} + \dfrac{\partial z}{\partial y}\dfrac{\partial y}{\partial \theta} = 3\sin\varphi(-\sin\theta) + 4\sin\varphi\cos\theta \end{cases}$$

$$\begin{tikzcd} \varphi \arrow[r, "x_\varphi"] \arrow[dr, "y_\varphi"] & x \arrow[dr, "z_x"] & \\ & & z \\ \theta \arrow[uur, "x_\theta"] \arrow[r, "y_\theta"'] & y \arrow[ur, "z_y"'] & \end{tikzcd}$$

$$\Rightarrow \begin{cases} z_\varphi = 3\left(\frac{\sqrt{3}}{2}\right)\left(\frac{1}{2}\right) + 4\left(\frac{\sqrt{3}}{2}\right)\left(\frac{\sqrt{3}}{2}\right) = \boxed{\frac{3\sqrt{3}}{4} + 3} \approx 4.30 \\ z_\theta = 3\left(\frac{1}{2}\right)\left(\frac{-\sqrt{3}}{2}\right) + 4\left(\frac{1}{2}\right)\left(\frac{1}{2}\right) = \boxed{1 - \frac{3\sqrt{3}}{4}} \approx -.299 \end{cases}$$

(2)

$$\begin{cases} 3 = z_x\left(\frac{\sqrt{3}}{2}\right)\left(\frac{1}{2}\right) + z_y\left(\frac{\sqrt{3}}{2}\right)\left(\frac{\sqrt{3}}{2}\right) \Rightarrow 12 = \sqrt{3}\,z_x + 3z_y \quad (E_1) \\ 4 = z_x\left(\frac{1}{2}\right)\left(\frac{-\sqrt{3}}{2}\right) + z_y\left(\frac{1}{2}\right)\left(\frac{1}{2}\right) \Rightarrow 16 = -\sqrt{3}\,z_x + z_y \quad (E_2) \end{cases}$$

$$(E_1 + E_2) \quad 28 = 4z_y \Rightarrow \boxed{z_y = 7}$$

$$(E_1 - 3E_2) \quad -36 = 4\sqrt{3}\,z_x \Rightarrow \boxed{z_x = -3\sqrt{3}} \approx -5.20$$

(3)

$$\begin{tikzcd} & x \arrow[dr, "z_x"] & \\ t \arrow[ur, "x_t"] \arrow[dr, "y_t"'] & & z \\ & y \arrow[ur, "z_y"'] & \end{tikzcd}$$

$$\Rightarrow \underbrace{\frac{dz}{dt}}_{7} = \frac{\partial z}{\partial x}\frac{dx}{dt} + \underbrace{\frac{\partial z}{\partial y}}_{-1}\underbrace{\frac{dy}{dt}}_{7}; \quad \frac{dx}{dt} = \underbrace{\underbrace{\cos(3t)}_{1/\sqrt{2}} - \underbrace{3t}_{\pi/4}\underbrace{\sin(3t)}_{1/\sqrt{2}}}_{\left(\frac{4}{4} - \frac{\pi}{4}\right)\frac{1}{\sqrt{2}}}, \quad t = \pi/12$$

$$\Rightarrow \underbrace{14}_{7-(-7)} = \frac{\partial z}{\partial x} \cdot \frac{4-\pi}{4} \cdot \frac{1}{\sqrt{2}} \Rightarrow z_x = \frac{\partial z}{\partial x} = \boxed{\frac{56\sqrt{2}}{4-\pi}} \approx 92.3$$

(1) Find the direction $\vec{u}$ parallel to $\langle 4, -3 \rangle$. HW25

Then compute $D_{\vec{u}} f(1,2)$ and $D_{\vec{u}} g(5,6)$

where $f(x,y) = \sin(xy)$ & $g(x,y) = \ln(r)$.

(2) Find the $\overbrace{\text{direction}}^{\text{(unit vector)}}$ $\vec{v}$ parallel to $\langle 2, -4, 5 \rangle$.

Then compute $D_{\vec{v}} f(3,2,1)$ and $D_{\vec{v}} g(-1,1,1)$

where $f(x,y,z) = \varphi$ and $g(x,y,z) = \theta$.

(In (1), (2), r & θ are polar coordinates and φ is a spherical coordinate.)

(2) Hint: $\frac{d}{du} \cos^{-1} u = \frac{-1}{\sqrt{1-u^2}}$

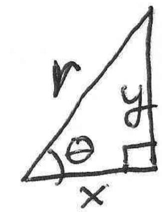
① $\boxed{\vec{u} = \left\langle \frac{4}{5}, -\frac{3}{5} \right\rangle}$. $\quad \vec{\nabla} f = \langle f_x, f_y \rangle = \langle y\cos(xy), x\cos(xy) \rangle$

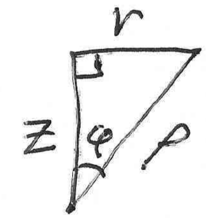
$$g = \ln\sqrt{x^2+y^2} = \frac{1}{2}\ln(x^2+y^2) \Rightarrow \vec{\nabla} g = \left\langle \frac{x}{x^2+y^2}, \frac{y}{x^2+y^2} \right\rangle$$

$$\vec{\nabla} f(1,2) = \langle 2,1 \rangle \cos 2 \,; \quad \vec{\nabla} g(5,6) = \langle 5,6 \rangle / 61.$$

$$D_{\vec{u}} f(1,2) = \vec{u} \cdot \vec{\nabla} f(1,2) = \boxed{\cos 2} \approx -.416$$

$$D_{\vec{u}} g(5,6) = \vec{u} \cdot \vec{\nabla} g(5,6) = \boxed{2/305} \approx .00656$$





② $\boxed{\vec{v} = \frac{1}{\sqrt{45}}\langle 2, -4, 5 \rangle. \quad \approx \langle .30, -.60, .75 \rangle}$

$$f = \varphi = \cos^{-1}\frac{z}{\rho} \left(= \cos^{-1}\left(z(x^2+y^2+z^2)^{-\frac{1}{2}} \right) \right)$$

$\pm = \text{sign}(y)$

$$g = \theta = \pm\cos^{-1}\frac{x}{r} \left(= \cos^{-1}\left(x(x^2+y^2)^{-\frac{1}{2}} \right) \right)$$

$$f_x = \frac{-1}{\sqrt{1-(z/\rho)^2}}\, z(-1)\rho^{-2}\left(\frac{x}{\rho}\right)^{\star} = \frac{xz}{\rho^2\sqrt{\rho^2-z^2}} = \frac{xz}{r\rho^2}; \quad f_y = \frac{yz}{r\rho^2};$$

$$f_z = \frac{-1}{\sqrt{1-(z/\rho)^2}}\left[\frac{1\rho - z(z/\rho)}{\rho^2}\right] = \frac{-1}{\sqrt{1-z^2/\rho^2}} \cdot \frac{\rho^2-z^2}{\rho^3} = \frac{-r^2}{r\rho^2} = -\frac{r}{\rho^2};$$

$\star$(I used $\partial\rho/\partial x = \frac{\partial}{\partial x}(x^2+y^2+z^2)^{1/2} = \frac{1}{2}(x^2+y^2+z^2)^{-\frac{1}{2}}(2x) = x/\rho$.)

$$g_x = \frac{\mp 1}{\sqrt{1-(x/r)^2}}\left[\frac{1r - x(x/r)}{r^2}\right] = \frac{\mp 1}{\sqrt{1-x^2/r^2}} \cdot \frac{r^2-x^2}{r^3} = \frac{\mp y^2}{\pm y\, r^2} = \frac{-y}{r^2};$$

$$g_y = \frac{\mp 1}{\sqrt{1-(x/r)^2}}\, x(-1)r^{-2}\left(\frac{y}{r}\right) = \frac{\pm yx}{r^2\sqrt{r^2-x^2}} = \frac{(\pm y)x}{r^2(\pm y)} = \frac{x}{r^2}; \quad g_z = 0$$

At $(x,y,z)=(3,2,1)$, $r=\sqrt{13}$ & $\rho=\sqrt{14}$. So,

HW 25

$$\vec{\nabla} f = \langle f_x, f_y, f_z\rangle = \left\langle \frac{3}{14\sqrt{13}}, \frac{2}{14\sqrt{13}}, -\frac{\sqrt{13}}{14}\right\rangle = \frac{\langle 3, 2, -13\rangle}{14\sqrt{13}}.$$

So, $D_{\vec{v}} f(3,2,1) = \vec{v}\cdot\vec{\nabla} f = \dfrac{6-8-65}{14\sqrt{13}\sqrt{45}} = \boxed{\dfrac{-67}{42\sqrt{65}}} \approx -.198$

At $(x,y,z)=(-1,1,1)$, $r=\sqrt{2}$. So, $\vec{\nabla} g = \left\langle -\frac{1}{2}, -\frac{1}{2}, 0\right\rangle$.

So, $D_{\vec{v}} g(-1,1,1) = \vec{v}\cdot\vec{\nabla} g = \dfrac{-1+2+0}{\sqrt{45}} = \boxed{\dfrac{1}{3\sqrt{5}}} \approx .149$

HW26

1. Find the line tangent to the curve $x^4 + xy + x^6 = 3$ at $(1,1)$.

2. Find the plane tangent to the surface $x^3 + y^3 + z^3 - 3xyz = 7$ at $(2,2,1)$.

3. Find the direction of fastest increase of $f(x,y) = x^{-2} + y^{-3}$ at $(1,1)$.

4. Find the direction of fastest decrease of $g(x,y,z) = x^2 + \left(\frac{y-z}{2}\right)^2 + \left(\frac{z-x}{3}\right)^2$ at $(-3,-4,-5)$.

HW 26

① $\vec{\nabla}(x^4+xy+x^6)=\langle 4x^3+y+6x^5,\ x\rangle=\langle 11, 1\rangle$

Tangent line: $\boxed{11(x-1)+1(y-1)=0}$

② $\vec{\nabla}(x^3+y^3+z^3-3xyz)=\langle 3(x^2-yz),\ 3(y^2-xz),\ 3(z^2-xy)\rangle$

$=3\langle 2, 2, -3\rangle$

Tangent plane: $\boxed{2(x-2)+2(y-2)-3(z-1)=0}$

($6(x-2)+6(y-2)-9(z-1)=0$ is the same plane.)

Actually, $(2,2,1)$ is not on the surface given.
I should have given $x^3+y^3+z^3-3xyz=5$. Sorry!

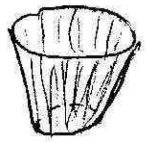
③ $\vec{\nabla}(x^{-2}+y^{-3})=\langle -2x^{-3}, -3y^{-4}\rangle=\langle -2,-3\rangle$ has direction $\boxed{-\langle 2,3\rangle/\sqrt{13}}\approx\langle -.55, -.83\rangle$

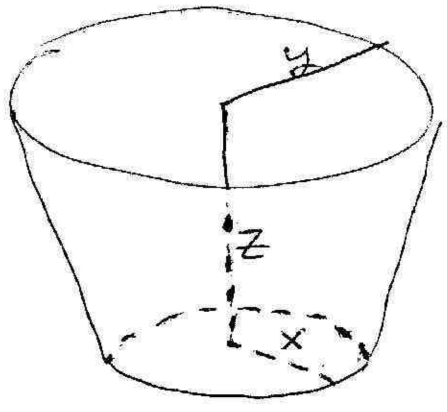
④ $-\vec{\nabla}\left(x^2+\left(\frac{y-z}{2}\right)^2+\left(\frac{z-x}{3}\right)^2\right)=-\langle 2x-\frac{2}{9}(z-x),\ \frac{2}{4}(y-z),\ \frac{2}{9}(z-x)-\frac{2}{4}(y-z)\rangle$
$=\langle 50/9, -1/2, 17/18\rangle$ has direction $\approx\boxed{\langle +.982, -.0884, +.167\rangle}$

HW27

① Find the point on the surface $z = x^2 + y^2$ closest to the point $(1, 2, -3)$.

Hint: minimize the squared distance from (x, y, x^2+y^2) to $(1, 2, -3)$.

② Consider a cup  that is a truncated cone with circlular base of radius x, (open) circular top with radius y, and height z. Given volume $V = 500\,cm^3$ (that's half a liter), find x, y, z that minimize the surface area S.



$$V = \frac{\pi z}{3}\left(x^2 + xy + y^2\right)$$

$$S = \pi x^2 + \pi(x+y)\sqrt{z^2 + (y-x)^2}$$

① $f(x,y) = (x-1)^2 + (y-2)^2 + (x^2+y^2+3)^2$ HW 27

$\vec{\nabla} f = 2\langle x-1+2x(x^2+y^2+3),\ y-2+2y(x^2+y^2+3)\rangle$

$= 2\langle 7x-1+2x(x^2+y^2),\ 7y-2+2y(x^2+y^2)\rangle.$

We must solve $\begin{cases} 1-7x = 2x(x^2+y^2) \\ 2-7y = 2y(x^2+y^2) \end{cases}$ (to solve $\vec{\nabla} f = \vec{0}$).

Use a calculator or computer to find an approximate solution: $(x,y) \approx (.139, .278)$. (There is only one (real) solution for this problem.) Since $z = x^2+y^2$,

$(x,y,z) \approx \boxed{(.139, .278, .0966).}$

① $f(x,y)=(x-1)^2+(y-2)^2+(x^2+y^2+3)^2$ HW 27

$$\vec{\nabla} f = 2\langle x-1+2x(x^2+y^2+3),\ y-2+2y(x^2+y^2+3)\rangle$$
$$=2\langle 7x-1+2x(x^2+y^2),\ 7y-2+2y(x^2+y^2)\rangle.$$

We must solve $\begin{cases} 1-7x = 2x(x^2+y^2) \\ 2-7y = 2y(x^2+y^2) \end{cases}$ (to solve $\vec{\nabla} f=\vec{0}$).

Use a calculator or computer to find an approximate solution: $(x,y)\approx(.139,.278)$. (There is only one (real) solution for this problem.) Since $z=x^2+y^2$,

$(x,y,z)\approx \boxed{(.139,.278,.0966).}$

If you try to solve $\vec{\nabla} f=\vec{0}$ by hand, you might notice $\frac{1}{x}-7=2(x^2+y^2)=\frac{2}{y}-7 \Rightarrow 2x=y$ and then substitute $2x$ for y to get $1-7x=2x(x^2+4x^2)=10x^3$. But then you have to solve a cubic equation in x...

HW 27

$$500 = \frac{\pi z}{3}(x^2+xy+y^2) \Rightarrow z = \frac{1500}{\pi(x^2+xy+y^2)}$$

(2)

$$\Rightarrow S = \pi x^2 + \pi(x+y)\sqrt{\underbrace{\left(\frac{1500}{\pi(x^2+xy+y^2)}\right)^2}_{\text{Use } z^2 \text{ for short}} + (y-x)^2}$$

$$\frac{d}{du}\sqrt{u} = \frac{1}{2\sqrt{u}}$$

$$\frac{\partial S}{\partial x} = 2\pi x + \pi\sqrt{z^2+(y-x)^2} + \pi(x+y)\frac{2z(dz/dx)+2(x-y)}{2\sqrt{z^2+(y-x)^2}}$$

$$\frac{\partial S}{\partial y} = 0 + \pi\sqrt{z^2+(y-x)^2} + \pi(x+y)\frac{2z(dz/dy)+2(y-x)}{2\sqrt{z^2+(y-x)^2}}$$

$$\frac{dz}{dx} = \frac{-1500(2x+y)}{\pi(x^2+xy+y^2)^2}\ ; \quad \frac{dz}{dy} = \frac{-1500(2y+x)}{\pi(x^2+xy+y^2)^2}$$

Note that you can compute $\frac{\partial S}{\partial x}$ & $\frac{\partial S}{\partial y}$ using a calculator or computer! You don't even need to see their giant formulas. You can store the formulas in variables and then use these for the next step.

We must solve $S_x = 0 = S_y$:

HW 27

$$0 = 2\pi x + \pi\sqrt{\frac{1500^2}{\pi^2(x^2+xy+y^2)^2} + (y-x)^2}$$

$$+ \frac{\pi(x+y)}{\sqrt{\frac{1500^2}{\pi^2(x^2+xy+y^2)^2} + (y-x)^2}}\left(\frac{-1500^2(2x+y)}{\pi^2(x^2+xy+y^2)^3} + x - y\right)$$

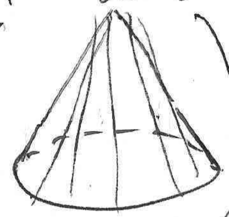
$$\& \; 0 = \pi\sqrt{\frac{1500^2}{\pi^2(x^2+xy+y^2)^2} + (y-x)^2}$$

$$+ \frac{\pi(x+y)}{\sqrt{\frac{1500^2}{\pi^2(x^2+xy+y^2)^2} + (y-x)^2}}\left(\frac{-1500^2(2y+x)}{\pi^2(x^2+xy+y^2)^3} + y - x\right)$$

Use technology to find an approximate solution (and to avoid having to type in such formulas manually).

I found two critical points (solutions to $S_x = S_y = 0$): HW 27

$(x,y) \approx \boxed{(3.628, 6.613)}$ & $(x,y) \approx (5.527, 0)$.

Common sense tells me the latter, a closed cone (), is not the cup I want. I also checked that the first critical point makes $S \approx 254.2$ & the "cone" solution makes $S \approx 383.8$.

$\boxed{\text{So, } x \approx 3.63\text{cm} \text{ \& } y \approx 6.61\text{cm minimize } S.}$