

① $\langle P, Q \rangle$	$Q_x = \frac{\partial Q}{\partial x}$	$P_y = \frac{\partial P}{\partial y}$	gradient "everywhere"?	HW 45
$\vec{A}$	$-y \sin(xy)$	$-x \sin(xy)$	no	
$\vec{B}$	$\cos(xy)$ $-xy \sin(xy)$	$\cos(xy)$ $-xy \sin(xy)$	maybe	
$\vec{C}$	$-y^2 \sin(xy)$	$-x^2 \sin(xy)$	no	
$\vec{D}$	$4x^3 - 6x^2y$	$4x^3 - 6x^2y$	maybe	
$\vec{E}$	$12xy^2 - 2y^3$	$4x^3 - 6x^2y$	no	

( $\vec{B}, \vec{D}$ ) have simply connected open domain  $\mathbb{R}^2$ ; so, as we learned on Day 46, we can upgrade "maybe" to "yes."

②  $\vec{F} = \vec{\nabla} \left( \int x^{-1} dx + \int y^{-2} dy \right) = \vec{\nabla} \left( \ln x - \frac{1}{y} \right)$  in the first quadrant  $\{(x, y) \mid x, y > 0\}$ , which contains  $I$  since  $\sqrt{1+t^3} \geq 1$  &  $\sqrt{t^5+1} \geq 1$  when  $t \geq 0$ . So,  $\int_I \vec{F} \cdot d\vec{r}$

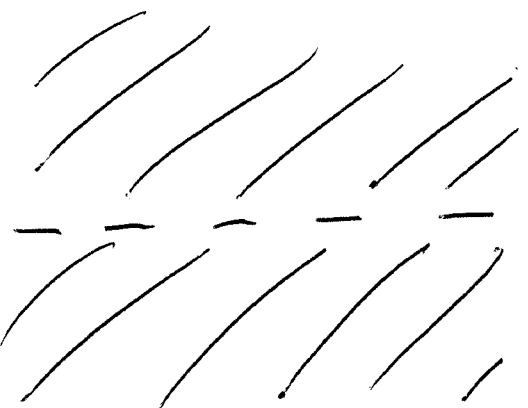
$$= \left( \ln x - \frac{1}{y} \right) \Big|_{t=0}^{t=1} = \left( \ln x - \frac{1}{y} \right) \Big|_{(x,y)=(1,1)}^{(x,y)=(\sqrt{2}, \sqrt{2})} = \boxed{\ln \sqrt{2} + 1 - \frac{1}{\sqrt{2}}}$$

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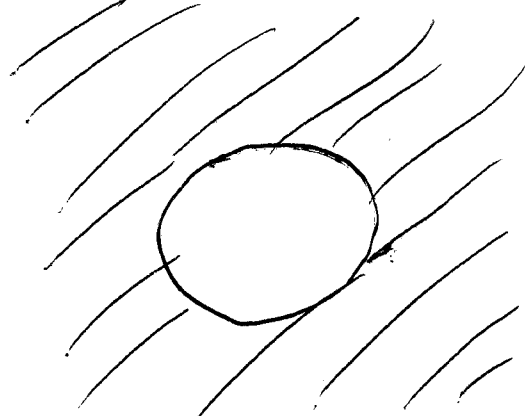
③  $\int_C \vec{F} \cdot d\vec{r} = (\ln x - \frac{1}{y}) \Big|_{(1,2)}^{(1,2)} = 0$

Sketches to motivate HW46 answers:

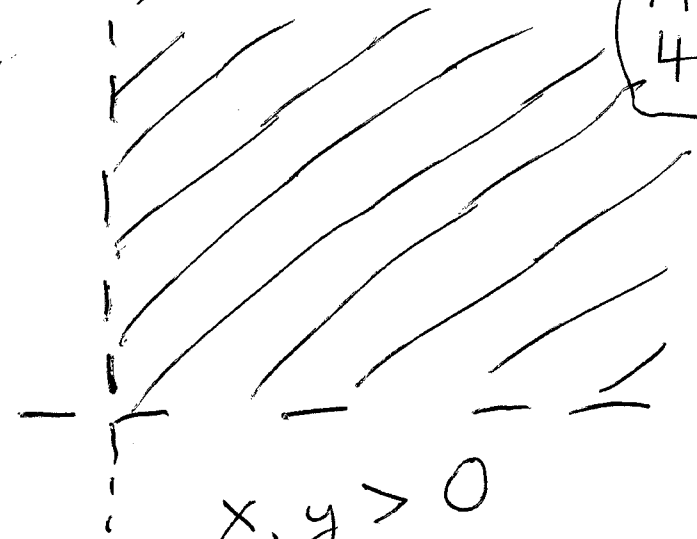
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$y < 0$



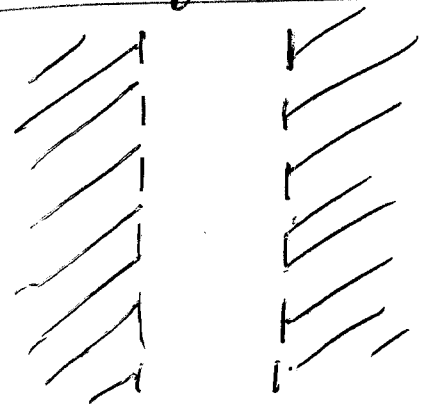
$y^2 > 5$



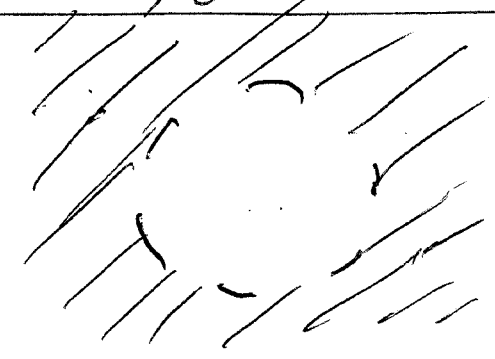
$x, y > 0$



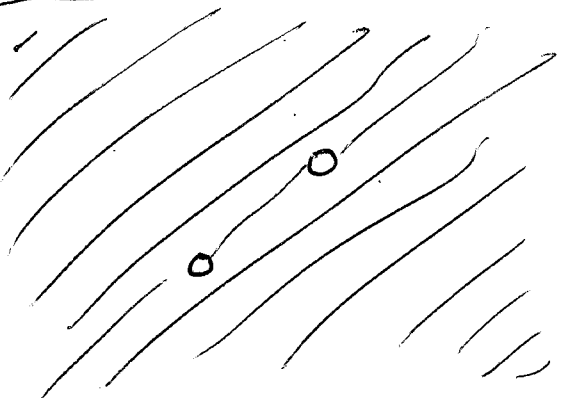
$r < 5$



$x^2 > 5$



$r > 5$



$(x, y) \neq (2, 3), (3, 4)$

Fill in the table with "yes" or "no." HW 46

Region	open?	connected?	simply connected?
$\{(x, y) \mid y \neq 0\}$	yes	no	no
$\{(x, y) \mid y^2 \geq 5\}$	no	yes	no
$\{(x, y) \mid x, y > 0\}$	yes	yes	yes
$\{(x, y) \mid r < 5\}$	yes	yes	yes
$\{(x, y) \mid x^2 > 5\}$	yes	no	no
$\{(x, y) \mid r > 5\}$	yes	yes	no
$\{(x, y) \mid (x, y) \neq (2, 3), (3, 4)\}$	yes	yes	no

It may help to sketch the region!

$$\textcircled{1} \quad f = \int P dx = 2x^3 + 5x^2y^4 - 3x + g(y)$$

$$\therefore Q = f_y = 0 + 20x^2y^3 + 0 + g'(y)$$

$$\therefore 1 = Q - 20x^2y^3 = g'(y) \quad \therefore y + c = g(y)$$

$$\therefore \boxed{f(x,y) = 2x^3 + 5x^2y^4 - 3x + y + c}$$

("∴") means "Therefore,"

$$\textcircled{2} \quad 7 = f(1,1) = 2 + 5 - 3 + 1 + c \Rightarrow c = 2$$

$$\therefore f(x,y) = 2x^3 + 5x^2y^4 - 3x + y + 2$$

$$\textcircled{3} \quad f = \int P dx = \int \frac{-y^2 dx}{x^2 \sqrt{x^2 + y^2}}$$

Let  $x = |y| \tan \alpha$  ( $y$  constant).

$$f = - \int \frac{|y| y^2 \sec^2 \alpha d\alpha}{y^2 \tan^2 \alpha |y| \sec \alpha}$$

$$\therefore \begin{cases} dx = |y| \sec^2 \alpha d\alpha \\ \sqrt{x^2 + y^2} = \sqrt{(1 + \tan^2 \alpha) y^2} \\ = |y| \sec \alpha \end{cases}$$

$$f = - \int \frac{\cos \alpha d\alpha}{\sin^2 \alpha} = - \int \frac{du}{u^2} = \frac{1}{u} + g(y)$$

(for  $-\frac{\pi}{2} < \alpha < \frac{\pi}{2}$ )

$$u = \sin \alpha = \sqrt{1 - \cos^2 \alpha} = \sqrt{1 - (\sec^2 \alpha)^{-1}} = \sqrt{1 - (1 + \tan^2 \alpha)^{-1}} = \dots$$

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$$\dots = \sqrt{(1 + \tan^2 \alpha - 1)(1 + \tan^2 \alpha)^{-1}} \stackrel{x > 0}{=} \tan \alpha / \sqrt{1 + \tan^2 \alpha}$$

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$$= |y| \tan \alpha / \sqrt{y^2 + y^2 \tan^2 \alpha} = x / \sqrt{y^2 + x^2} = x / r.$$

$$\therefore f(x, y) = \frac{1}{u} + g(y) = \frac{r}{x} + g(y).$$

$$\therefore \frac{y}{xr} = Q = f_y = g'(y) + \frac{\partial}{\partial y} \left( \frac{r}{x} \right).$$

$$\frac{\partial}{\partial y} \left( \frac{r}{x} \right) = \frac{1}{x} \frac{\partial}{\partial y} (x^2 + y^2)^{\frac{1}{2}} = \frac{1}{2x} (x^2 + y^2)^{\frac{1}{2} - 1} (2y) = \frac{y}{x\sqrt{x^2 + y^2}} = \frac{y}{xr}$$

$$\therefore g'(y) = 0. \quad \text{So, } f(x, y) = \frac{r}{x} + c = \frac{\sqrt{x^2 + y^2}}{x} + c$$

$$\therefore 1 = f(1, 0) = \frac{\sqrt{1^2 + 0^2}}{1} + c \quad \therefore c = 0.$$

$$\therefore f(x, y) = \frac{\sqrt{x^2 + y^2}}{x} = \frac{r}{x} = \sec \theta$$

④ Use the substitution  $x = -|y| \tan \alpha$  this time.

Still with  $u = \sin \alpha$ ,  $f(x,y) = \frac{-1}{u} + g(y) = \frac{r}{x} + g(y) = \frac{\sqrt{x^2+y^2}}{x} + c$

like in ③.  $5 = f(-1,0) = (\sqrt{(-1)^2+0^2}/(-1)) + c \Rightarrow c = 6.$

So,  $f(x,y) = r/x + 6 = \sqrt{x^2+y^2}/x + 6 = \sec \theta + 6.$

⑤  $f = \int P dx = \int \frac{x dx}{(x^2+y^2)^{3/2}} = \int \frac{du/2}{u^{3/2}} = \frac{1}{2} \int u^{-3/2} du = -u^{-1/2} + g(y)$

$u = x^2 + y^2$  (y constant)  
 $du = 2x dx$

$= -\frac{1}{r} + g(y)$ .  $\therefore y/r^3 = Q = f_y = g'(y) - \frac{\partial}{\partial y} (x^2+y^2)^{-1/2}$

$= g'(y) - (-\frac{1}{2})(x^2+y^2)^{-3/2} (2y) = g'(y) + y/r^3$ .  $\therefore g'(y) = 0.$

$\therefore f = -\frac{1}{r} + c$ . These solutions are well-defined &

differentiable everywhere except (0,0). So, yes,

$F(x,y) = \frac{-1}{\sqrt{x^2+y^2}}$  (choosing  $c=0$ , for example) is a solution

to  $\vec{\nabla} f = \langle x,y \rangle / r^3$  on all of  $\mathbb{R}^2 - \{(0,0)\}$ .

$$\textcircled{1} \iint_D (Q_x - P_y) dA = \int_{\partial D} (P dx + Q dy) \quad (\text{Green's Thm.}) \quad \boxed{\text{HW } 48}$$

P	Q	$Q_x - P_y$
0	x	1 - 0
0	$x^2/2$	x - 0
0	xy	y - 0

$$x_{cm} = \frac{\iint_D x dA}{\iint_D 1 dA} = \frac{\int_{\partial D} x^2 dy / 2}{\int_{\partial D} x dy}$$

$$y_{cm} = \frac{\iint_D y dA}{\iint_D 1 dA} = \frac{\int_{\partial D} xy dy}{\int_{\partial D} x dy}$$

$$\int_{\partial D} x dy = \int_0^1 (t - t^3)(2t - 1) dt = 1/60$$

$$\frac{1}{2} \int_{\partial D} x^2 dy = \frac{1}{2} \int_0^1 (t - t^3)^2 (2t - 1) dt = 1/280$$

$$\int_{\partial D} xy dy = \int_0^1 (t - t^3)(t^2 - t)(2t - 1) dt = -1/420$$

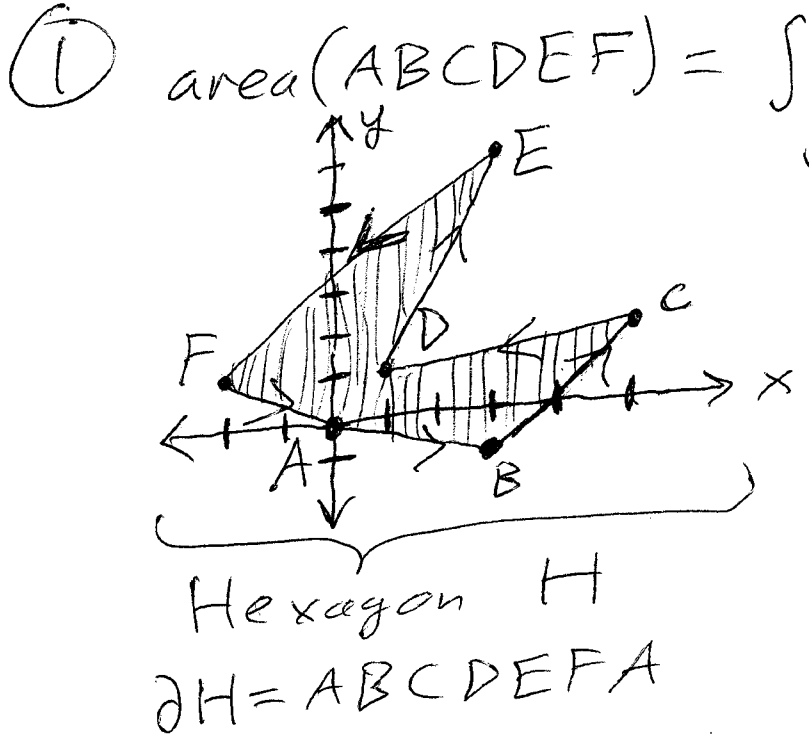
$$\Rightarrow \begin{cases} x_{cm} = 3/14 \\ y_{cm} = -1/7 \end{cases}$$

$$\textcircled{2} \begin{matrix} (7,2) & \leftarrow & (7,4) \\ \downarrow D & & \uparrow C = \partial D \\ (5,2) & \rightarrow & (7,2) \end{matrix} \quad \iint_D = \int_{x=5}^{x=7} \int_{y=2}^{y=4}$$

$$\int_C \left( \frac{dx}{y} - \frac{dy}{x} \right)$$

$$= \int_{\partial D} (y^{-1} dx - x^{-1} dy) = \iint_D \left( (-x^{-1})_x - (y^{-1})_y \right) dA$$

$$= \int_5^7 \left( \int_2^4 (x^{-2} + y^{-2}) dy \right) dx = \boxed{\frac{43}{70}}$$



counterclockwise boundary loop = AB + BC + CD + DE + EF + FA

segment	AB	BC	CD	DE	EF	FA
x	3t	3+2t	5-4t	1+2t	3-5t	-2+2t
dy	-dt	3dt	-dt	5dt	-5dt	-dt

parametrizations each from t=0 to t=1.

$$\text{area} = \int_0^1 (-3t + 9 + 6t - 5 + 4t + 5 + 10t - 15 + 25t + 2 - 2t) dt$$

$$= \int_0^1 (-4 + 40t) dt = -4 + 20 = \boxed{16}$$

②  $x_{cm} = \frac{1}{16} \iint_H x dA = \frac{1}{16} \int_{\partial H} \frac{x^2}{2} dy = \frac{1}{32} \int_0^1 [(3t)^2(-1) + (3+2t)^2(3) + (5-4t)^2(-1) + (1+2t)^2(5) + (3-5t)^2(-5) + (-2+2t)^2(-1)] dt = \boxed{\frac{133}{96}}$

$y_{cm} = \frac{1}{16} \iint_H y dA = \frac{1}{16} \int_{\partial H} xy dy$

segment	AB	BC	CD	DE	EF	FA
y	-t	3t-1	2-t	1+5t	6-5t	1-t

$$= \frac{1}{16} \int_0^1 [(3t)(-t)(-1) + (3+2t)(3t-1)(3) + (5-4t)(2-t)(-1) + \dots] dt = \boxed{\frac{73}{48}}$$



$$\iint_E xy \, dx \, dy = \iint_E \left( \frac{\partial}{\partial x} (x^2 y / 2) - \frac{\partial}{\partial y} (0) \right) dA \quad \left. \begin{array}{l} \text{HW} \\ 49 \end{array} \right\}$$

$$= \int_{\partial E} \left( 0 \, dx + \frac{1}{2} x^2 y \, dy \right)$$

$$= \left( \int_I x^2 y \, dy + \int_J x^2 y \, dy + \int_K x^2 y \, dy \right) / 2$$

$$= \frac{1}{2} \int_0^{2\pi} \left( \left[ (3 + (6 + \sin(7t)) \cos(t))^2 \right] \left[ (4 + 2 \sin(7t)) \sin(t) \right] \left[ 14 \cos(7t) \sin(t) \right] \right. \\ \left. + (4 + 2 \sin(7t)) \cos(t) \right)$$

$$+ \left[ 1 + 2 \cos(t) \right]^2 \left[ -\sin(t) \right] \left[ -\cos(t) \right]$$

$$+ \left[ 5 + \cos(t) \right]^2 \left[ 1 - 2 \sin(t) \right] \left[ -2 \cos(t) \right] \Big) dt$$

$$= \boxed{-20\pi}$$