Continuing from Day 23: Here's where we left off: $z = r\theta$. $\frac{\partial z}{\partial x} = ?$ at $\begin{cases} r = 7 \\ \theta = \pi/4 \end{cases}$.

$$\underbrace{\frac{\partial z}{\partial r}}_{\theta} dr + \underbrace{\frac{\partial z}{\partial \theta}}_{r} d\theta \overset{}{\underset{(D)}{=}} dz \overset{(A)}{=} \frac{\partial z}{\partial x} dx + \frac{\partial z}{\partial y} dy$$

$x = r\cos\theta$

$$dx \overset{(B)}{=} \frac{\partial x}{\partial r} dr + \frac{\partial x}{\partial \theta} d\theta = \cos\theta\, dr - r\sin\theta\, d\theta$$

$$dy \overset{(C)}{=} \frac{\partial y}{\partial r} dr + \frac{\partial y}{\partial \theta} d\theta = \sin\theta\, dr + r\cos\theta\, d\theta$$

$y = r\sin\theta$

Now we continue: Substitute $r=7$, $\theta = \pi/4$ into (B), (C), (D):

(B') $dx = dr/\sqrt{2} - 7\,d\theta/\sqrt{2}$

(C') $dy = dr/\sqrt{2} + 7\,d\theta/\sqrt{2}$

(D') $dz = \frac{\pi}{4} dr + 7\,d\theta$

Next, combine (A) with (B'), (C'), (D'):

$$\frac{\partial z}{\partial x}\underbrace{\left(\frac{dr - 7\,d\theta}{\sqrt{2}}\right)}_{dx} + \frac{\partial z}{\partial y}\underbrace{\left(\frac{dr + 7\,d\theta}{\sqrt{2}}\right)}_{dy} = dz \underset{(E)}{=} \frac{\pi}{4}\,dr + 7\,d\theta$$

Now group by dr & $d\theta$

(F) $$\frac{1}{\sqrt{2}}\left(\frac{\partial z}{\partial x} + \frac{\partial z}{\partial y}\right)dr + \frac{7}{\sqrt{2}}\left(\frac{\partial z}{\partial y} - \frac{\partial z}{\partial x}\right)d\theta$$

Then equate the dr & $d\theta$ terms from (E), (F):

$$\frac{1}{\sqrt{2}}\left(\frac{\partial z}{\partial x} + \frac{\partial z}{\partial y}\right) = \frac{\pi}{4} \quad \& \quad \frac{7}{\sqrt{2}}\left(\frac{\partial z}{\partial y} - \frac{\partial z}{\partial x}\right) = 7.$$

Finally, solve for $z_x = \frac{\partial z}{\partial x}$:

$$\begin{cases} z_x + z_y = \pi\sqrt{2}/4 \\ z_y - z_x = \sqrt{2} \end{cases} \underset{\text{subtract equations}}{\Rightarrow} 2z_x = \sqrt{2}\left(\frac{\pi}{4} - 1\right) \Rightarrow z_x = \frac{\sqrt{2}}{2}\left(\frac{\pi}{4} - 1\right) \approx \boxed{-0.152}$$

☆ Speeding up the solution with more chain rules: ☆

$$\begin{tikzcd} r \arrow[r] \arrow[ddr] & x \arrow[dr] & \\ & & z \\ \theta \arrow[r] \arrow[uur] & y \arrow[ur] & \end{tikzcd}$$

$$\frac{\partial z}{\partial r} = \frac{\partial z}{\partial x}\frac{\partial x}{\partial r} + \frac{\partial z}{\partial y}\frac{\partial y}{\partial r} \qquad \text{G}$$

$$\frac{\partial z}{\partial \theta} = \frac{\partial z}{\partial x}\frac{\partial x}{\partial \theta} + \frac{\partial z}{\partial y}\frac{\partial y}{\partial \theta} \qquad \text{H}$$

$$z = r\theta \Rightarrow \frac{\partial z}{\partial r} = \theta = \frac{\pi}{4} \quad \& \quad \frac{\partial z}{\partial \theta} = r = 7$$

$\boxed{r = 7 \ \& \ \theta = \pi/4 \text{ were given.}}$

$$x = r\cos\theta \Rightarrow \frac{\partial x}{\partial r} = \cos\theta = \frac{1}{\sqrt{2}} \quad \& \quad \frac{\partial x}{\partial \theta} = -r\sin\theta = -\frac{7}{\sqrt{2}}$$

$$y = r\sin\theta \Rightarrow \frac{\partial y}{\partial r} = \sin\theta = \frac{1}{\sqrt{2}} \quad \& \quad \frac{\partial y}{\partial \theta} = r\cos\theta = +\frac{7}{\sqrt{2}}$$

Substitute the #s into G & H:

$$\left.\begin{aligned} \text{G}' \quad & \frac{\pi}{4} = z_x/\sqrt{2} + z_y/\sqrt{2} \\ \text{H}' \quad & 7 = z_x \cdot \frac{-7}{\sqrt{2}} + z_y \cdot \frac{7}{\sqrt{2}} \end{aligned}\right\} \Longrightarrow \left\{\begin{aligned} \sqrt{2}\,\pi/4 &= z_x + z_y \\ \sqrt{2} &= z_y - z_x \end{aligned}\right.$$

$$\Rightarrow 2z_x = \sqrt{2}\left(\frac{\pi}{4} - 1\right) \Rightarrow \boxed{z_x \approx -.152}$$

Alternative solution using matrices:

$$\begin{bmatrix} \dfrac{\partial z}{\partial r} \\[2ex] \dfrac{\partial z}{\partial \theta} \end{bmatrix} = \begin{bmatrix} \dfrac{\partial x}{\partial r} & \dfrac{\partial y}{\partial r} \\[2ex] \dfrac{\partial x}{\partial \theta} & \dfrac{\partial y}{\partial \theta} \end{bmatrix} \begin{bmatrix} \dfrac{\partial z}{\partial x} \\[2ex] \dfrac{\partial z}{\partial y} \end{bmatrix} \Rightarrow \begin{bmatrix} \theta \\ r \end{bmatrix} = \begin{bmatrix} \cos\theta & \sin\theta \\ -r\sin\theta & r\cos\theta \end{bmatrix} \begin{bmatrix} z_x \\ z_y \end{bmatrix}$$

Using $z = r\theta$ & $\begin{cases} x = r\cos\theta \\ y = r\sin\theta \end{cases}$

$$\Rightarrow \begin{bmatrix} z_x \\ z_y \end{bmatrix} = \begin{bmatrix} \cos\theta & \sin\theta \\ -r\sin\theta & r\cos\theta \end{bmatrix}^{-1} \begin{bmatrix} \theta \\ r \end{bmatrix}$$

using $\begin{cases} r = 7 \\ \theta = \pi/4 \end{cases}$

$$= \begin{bmatrix} 1/\sqrt{2} & 1/\sqrt{2} \\ -7/\sqrt{2} & 7/\sqrt{2} \end{bmatrix}^{-1} \begin{bmatrix} \pi/4 \\ 7 \end{bmatrix}$$

$$= \underbrace{\left(\frac{1}{\frac{1}{\sqrt{2}}\cdot\frac{7}{\sqrt{2}} - \frac{1}{\sqrt{2}}\cdot\frac{-7}{\sqrt{2}}} \right)}_{1/(7/2 + 7/2) = 1/7} \overbrace{\begin{bmatrix} 7/\sqrt{2} & -1/\sqrt{2} \\ 7/\sqrt{2} & 1/\sqrt{2} \end{bmatrix} \begin{bmatrix} \pi/4 \\ 7 \end{bmatrix}}^{= \begin{bmatrix} (7/\sqrt{2})(\pi/4 - 1) \\ (7/\sqrt{2})(\pi/4 + 1) \end{bmatrix}} \approx \begin{bmatrix} -.152 \\ 1.262 \end{bmatrix}$$