# Part I

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & -1 \end{bmatrix} :$$ Rotation (of 180° around x-axis)

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 1 \end{bmatrix} :$$ Reflection (about the plane $y=0$)

$$\begin{bmatrix} 1 & 5 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} :$$ skew

$$\begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix} :$$ Rotation (90° counterclockwise)

$$\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} :$$ Reflection (about line $x=y$)

$$\begin{bmatrix} 3 & 0 \\ 0 & 4 \end{bmatrix} :$$ scale change

$\begin{bmatrix} \sqrt{3}/2 & -1/2 \\ 1/2 & \sqrt{3}/2 \end{bmatrix}$ : rotation (30° counter-clockwise)

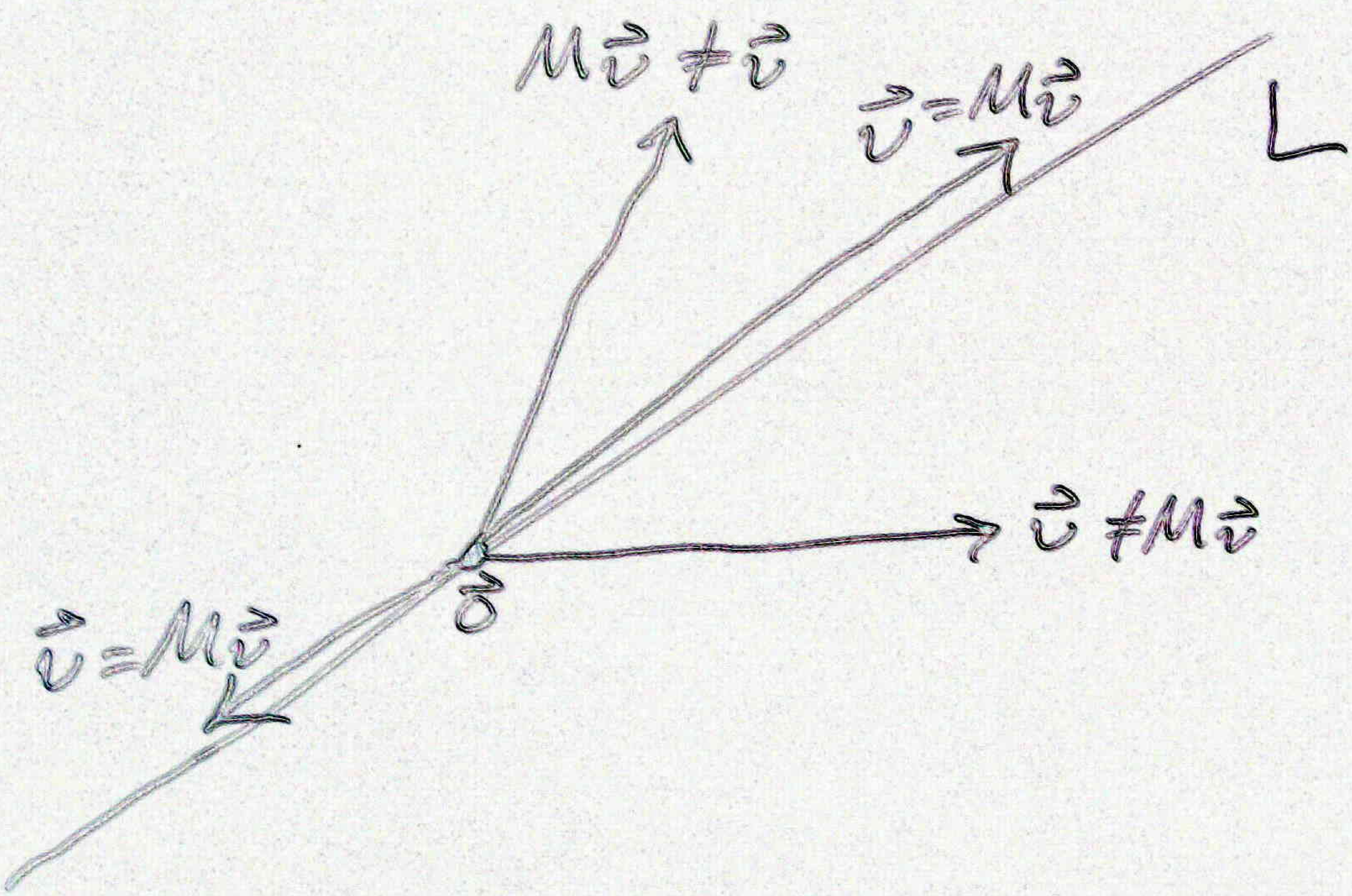
$\begin{bmatrix} 1 & 1 \\ 1 & 0 \end{bmatrix}$ : none of the above

$\begin{bmatrix} -1 & 1 \\ 1 & -1 \end{bmatrix}$ : none of the above

$\begin{bmatrix} 3/5 & 4/5 \\ 4/5 & -1/5 \end{bmatrix}$ : none of the above

One way to prove the "none of the aboves": They don't have the $\begin{bmatrix} \cos(\theta) & -\sin(\theta) \\ \sin(\theta) & \cos(\theta) \end{bmatrix}$ form; they don't have skew $\begin{bmatrix} 1 & 0 \\ a & 1 \end{bmatrix}$ or $\begin{bmatrix} 1 & a \\ 0 & 1 \end{bmatrix}$ form; they don't have the scale change $\begin{bmatrix} a & 0 \\ 0 & b \end{bmatrix}$ (with $a,b>0$) form. Also, they are not reflections about lines because in all three cases, letting $M$ be the matrix, $M\vec{v} = \vec{v} \iff \vec{v} \in \ker(M-I)$ $\iff \vec{v} = \vec{0}$ can be shown using G-J elimination.

A reflection $M$ about a line $L$ satisfies $M\vec{v} = \vec{v}$ for exactly those $\vec{v}$ in $L$. $\{\vec{0}\}$ is not a line.



## Part II


$$\begin{array}{ccc} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & -1 \\ 0 & 1 & 0 \end{bmatrix} \\ \vec{e}_1 \quad \vec{e}_3 \quad -\vec{e}_2 \end{array}$$

rotates $\mathbb{R}^3$ $90°$ ccw around the $+x$-axis (sending $\vec{e}_2$ to $\vec{e}_3$ & $\vec{e}_3$ to $-\vec{e}_2$ & $\vec{e}_1$ to $\vec{e}_1$ )

(Note: $\begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix} = \begin{bmatrix} \cos\frac{\pi}{2} & -\sin\frac{\pi}{2} \\ \sin\frac{\pi}{2} & \cos\frac{\pi}{2} \end{bmatrix}$)

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$$P=\{(x,y,z): x+2y+3z=0\}=\ker([1\ 2\ 3])$$

$$=\left\{\begin{bmatrix} x \\ y \\ z \end{bmatrix} : \begin{matrix} x=-2y-3z \\ y=y \\ z=z \end{matrix}\right\} = \text{span}\left\{\underset{\vec{u}_1}{\begin{bmatrix} -2 \\ 1 \\ 0 \end{bmatrix}}, \underset{\vec{u}_2}{\begin{bmatrix} -3 \\ 0 \\ 1 \end{bmatrix}}\right\}$$

Let $\vec{v}_1=\vec{u}_1$ & $\vec{v}_2=\vec{u}_2-\dfrac{\vec{v}_1\cdot\vec{u}_2}{\vec{v}_1\cdot\vec{v}_1}\vec{v}_1$ to get an orthogonal basis of $P$.

$$\vec{v}_2=\begin{bmatrix} -3 \\ 0 \\ 1 \end{bmatrix}-\frac{6}{5}\begin{bmatrix} -2 \\ 1 \\ 0 \end{bmatrix}=\begin{bmatrix} 0.6 \\ -1.2 \\ 1 \end{bmatrix}.$$

The reflection $R$ about $P$ is given by

$$R\vec{w}=2\,\text{proj}_P\vec{w}-\vec{w}=2(\text{proj}_{\vec{v}_1}\vec{w}+\text{proj}_{\vec{v}_2}\vec{w})-\vec{w}$$

$$=2\left(\frac{\vec{v}_1\cdot\vec{w}}{\vec{v}_1\cdot\vec{v}_1}\right)\vec{v}_1+2\left(\frac{\vec{v}_2\cdot\vec{w}}{\vec{v}_2\cdot\vec{v}_2}\right)\vec{v}_2-\vec{w}$$

$$R\vec{e}_1 = 2\left(\frac{-2}{5}\right)\begin{bmatrix} -2 \\ 1 \\ 0 \end{bmatrix} + 2\left(\frac{0.6}{2.8}\right)\begin{bmatrix} 0.6 \\ -1.2 \\ 1 \end{bmatrix} - \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$$

$$= \begin{bmatrix} 6/7 \\ -46/35 \\ 3/7 \end{bmatrix}$$

$$R\vec{e}_2 = 2\left(\frac{1}{5}\right)\begin{bmatrix} -2 \\ 1 \\ 0 \end{bmatrix} + 2\left(\frac{-1.2}{2.8}\right)\begin{bmatrix} 0.6 \\ -1.2 \\ 1 \end{bmatrix} - \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}$$

$$= \begin{bmatrix} 2/7 \\ -13/35 \\ -6/7 \end{bmatrix}$$

$$R\vec{e}_3 = 2\left(\frac{0}{5}\right)\begin{bmatrix} -2 \\ 1 \\ 0 \end{bmatrix} + 2\left(\frac{1}{2.8}\right)\begin{bmatrix} 0.6 \\ -1.2 \\ 1 \end{bmatrix} - \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$$

$$= \begin{bmatrix} 3/7 \\ -6/7 \\ 2/7 \end{bmatrix}$$

The matrix $\begin{bmatrix} R\vec{e}_1 & R\vec{e}_2 & R\vec{e}_3 \end{bmatrix}$ of R

is $\begin{bmatrix} 6/7 & 2/7 & 3/7 \\ -46/35 & -13/35 & -6/7 \\ 3/7 & -6/7 & 2/7 \end{bmatrix}$

45° ccw rotation about +z-axis:

$$\begin{bmatrix} \cos \pi/4 & -\sin \pi/4 & 0 \\ \sin \pi/4 & \cos \pi/4 & 0 \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 1/\sqrt{2} & -1/\sqrt{2} & 0 \\ 1/\sqrt{2} & 1/\sqrt{2} & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

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If $R$ reflects $\mathbb{R}^2$ about $L = \operatorname{span}\left\{\begin{bmatrix} 3 \\ -4 \end{bmatrix}\right\}$,

then $R\vec{w} = 2\left(\frac{\vec{v}\cdot\vec{w}}{\vec{v}\cdot\vec{v}}\right)\vec{v} - \vec{w}$ where $\vec{v} = \begin{bmatrix} 3 \\ -4 \end{bmatrix}$.

$$R\vec{e}_1 = 2\left(\frac{3}{25}\right)\begin{bmatrix} 3 \\ -4 \end{bmatrix} - \begin{bmatrix} 1 \\ 0 \end{bmatrix} = \begin{bmatrix} -7/25 \\ -24/25 \end{bmatrix}$$

$$R\vec{e}_2 = 2\left(\frac{-4}{25}\right)\begin{bmatrix} 3 \\ -4 \end{bmatrix} - \begin{bmatrix} 0 \\ 1 \end{bmatrix} = \begin{bmatrix} -24/25 \\ 7/25 \end{bmatrix}$$

$$R = \begin{bmatrix} R\vec{e}_1 & R\vec{e}_2 \end{bmatrix} = \begin{bmatrix} -7/25 & -24/25 \\ -24/25 & +7/25 \end{bmatrix}$$

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120° ccw rotation about +y-axis:

$$\begin{bmatrix} \cos 2\pi/3 & 0 & \sin 2\pi/3 \\ 0 & 1 & 0 \\ -\sin 2\pi/3 & 0 & \cos 2\pi/3 \end{bmatrix} = \begin{bmatrix} -1/2 & 0 & +\sqrt{3}/2 \\ 0 & +1 & 0 \\ -\sqrt{3}/2 & 0 & -1/2 \end{bmatrix}$$