3.1) pivots col's $\underbrace{1, 3, 4, 6}_{}$ $\Rightarrow$ free vars x_2, x_5

dependent var's x_1, x_3, x_4, x_6

Row 1: $\underbrace{x_1}_{\text{dep.}} + \underbrace{x_2 - x_5}_{\text{free}} = 4 \Rightarrow \left\{ \begin{aligned} x_1 &= -x_2 + x_5 + 4 \\ x_3 &= -2x_5 + 5 \\ x_4 &= -3x_5 + 6 \\ x_6 &= 7 \end{aligned} \right.$

Solution set is:

$$\left\{ \begin{bmatrix} -x_2 + x_5 + 4 \\ x_2 \\ -2x_5 + 5 \\ -3x_5 + 6 \\ x_5 \\ 7 \end{bmatrix} \middle| \; x_2, x_5 \in \mathbb{C} \right\}$$

3.2) 3 examples:

	1	2	3
x_2	0	1	0
x_5	0	0	1

$\Rightarrow$

$$\overset{1}{\begin{bmatrix} 4 \\ 0 \\ 5 \\ 6 \\ 0 \\ 7 \end{bmatrix}}, \quad \overset{2}{\begin{bmatrix} 3 \\ 1 \\ 5 \\ 6 \\ 0 \\ 7 \end{bmatrix}}, \quad \overset{3}{\begin{bmatrix} 5 \\ 0 \\ 3 \\ 3 \\ 1 \\ 7 \end{bmatrix}}$$

$\underline{4.1}$ $LS\left(A, \begin{bmatrix} 2 \\ 3 \\ 4 \\ 5 \end{bmatrix}\right)$ has no solutions <u>or</u> infinitely many solutions (of the form

$$\begin{bmatrix} ? \\ ? \\ ? \\ ? \end{bmatrix} + \alpha \begin{bmatrix} 5 \\ 0 \\ 0 \\ 1 \end{bmatrix} \text{ for } \alpha \in \mathbb{C} \text{ arbitrary}).$$

Example with no solutions:

$$\left[\begin{array}{cccc|c} \multicolumn{4}{c|}{\overbrace{\hphantom{1 \quad 0 \quad 0 \quad -5}}^{A}} & \\ 1 & 0 & 0 & -5 & 2 \\ 0 & 0 & 0 & 0 & 3 \\ 0 & 0 & 0 & 0 & 4 \\ 0 & 0 & 0 & 0 & 5 \end{array}\right]$$

Example with ∞ solutions:

RREF

$$\underbrace{\left[\begin{array}{cccc|c} 1 & 0 & 0 & -5 & 2 \\ 0 & 1 & 0 & 0 & 3 \\ 0 & 0 & 1 & 0 & 4 \\ 2.5 & 0 & 0 & -12.5 & 5 \end{array}\right]}_{A} \xleftarrow[\text{row op's}]{2.5R_1 + R_4} \left[\begin{array}{cccc|c} 1 & 0 & 0 & -5 & 2 \\ 0 & 1 & 0 & 0 & 3 \\ 0 & 0 & 1 & 0 & 4 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right]$$

4.2

$$N(A) = \left\{ x_2 \begin{bmatrix} -4 \\ 1 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix} + x_4 \begin{bmatrix} 1 \\ 0 \\ -2 \\ 1 \\ 0 \\ 0 \\ 0 \end{bmatrix} + x_7 \begin{bmatrix} -5 \\ 0 \\ 0 \\ 0 \\ -3 \\ 3 \\ 1 \end{bmatrix} \;\middle|\; x_2, x_4, x_7 \in \mathbb{C} \right\}$$

(Each column vector should have 7 rows.)

4.3

3 $\quad 20 \begin{bmatrix} 4 \\ -3 \end{bmatrix} - 7 \begin{bmatrix} 1 \\ 2 \end{bmatrix} = ? \quad \begin{bmatrix} 20 \cdot 4 - 7 \cdot 1 \\ 20(-3) - 7 \cdot 2 \end{bmatrix} = \begin{bmatrix} 73 \\ -74 \end{bmatrix}$

Day 5

(1) Assuming $\begin{bmatrix} 0 \\ 1 \\ 2 \end{bmatrix} \in N(A)$ and

$\begin{bmatrix} 5 \\ 1 \\ 7 \end{bmatrix}$ is a solution to $LS(A, \vec{b})$,

find three other solutions to $LS(A, \vec{b})$.

(2) If the solution set for $LS(A, \vec{b}) \neq \phi$, then $\vec{b}$ is a linear combination of columns of A.

(3) If the solution set for $LS(A, \vec{b}) = \phi$, then $\vec{b}$ is not a linear combination of columns of A.

~~(4)~~ For (2) & (3), fill in the blank.

5.1) If $\vec{u}$ solves $LS(A, \vec{b})$
& $\vec{v}$ solves $LS(A, \vec{c})$,
then $\vec{u}+\vec{v}$ solves $LS(A, \vec{b}+\vec{c})$
& $\alpha\vec{u}$ solves $LS(A, \alpha\vec{b})$.

$$\begin{bmatrix} 5 \\ 1 \\ 7 \end{bmatrix} + \alpha \begin{bmatrix} 0 \\ 1 \\ 2 \end{bmatrix} \text{ solves } LS(A, \overbrace{\vec{b} + \underbrace{\alpha\vec{0}}_{\vec{0}}}^{\vec{b}})$$

$\alpha = 1, 2, 3$: $\begin{bmatrix} 5 \\ 2 \\ 9 \end{bmatrix}, \begin{bmatrix} 5 \\ 3 \\ 11 \end{bmatrix}, \begin{bmatrix} 5 \\ 4 \\ 13 \end{bmatrix}$

6.1)

$$A = \begin{bmatrix} \vec{v}_1 & \vec{v}_2 & \vec{v}_3 & \vec{v}_4 \\ & & & \end{bmatrix} \xrightarrow[\text{op's}]{\text{row}} R = \begin{bmatrix} 1 & 0 & 3 & 5 \\ 0 & 1 & -1 & -7 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

A & R are row-equivalent $\Rightarrow N(A) = N(R)$

$$\Rightarrow \left(\begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \end{bmatrix} x_1 + \begin{bmatrix} 0 \\ 1 \\ 0 \\ 0 \end{bmatrix} x_2 + \begin{bmatrix} 3 \\ -1 \\ 0 \\ 0 \end{bmatrix} x_3 + \begin{bmatrix} 5 \\ -7 \\ 0 \\ 0 \end{bmatrix} x_4 = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix} \right. \ldots$$

6.1

... has same solution set as

$x_1\vec{v}_1 + x_2\vec{v}_2 + x_3\vec{v}_3 + x_4\vec{v}_4 = \vec{0}$

We want $x\vec{v}_1 + y\vec{v}_2 + z\vec{v}_3 = \vec{v}_4$.

$x = x_1; \quad y = x_2; \quad z = x_3 = 0; \quad x_4 = -1$

free vars

$x_3 = $ whatever

$x_4 = -1$

From R: $x_1 + 5\overset{(-1)}{x_4} = 0$

$x_2 - 7\underbrace{(x_4)}_{-1} = 0$

$x_3 = 0$

$\Longrightarrow$

$x_1 = x = 5$

$x_2 = y = -7$

$z = x_3 = 0$

$x_4 = -1$

$$\boxed{\vec{v}_4 = 5\vec{v}_1 - 7\vec{v}_2 + 0\vec{v}_3}$$

Compare to R

$\boxed{6.1}$ If $\overset{z}{\overset{\|}{x_3}} = 1$ is chosen, then from R:

$$x_1 + 3x_3 + 5x_4 = 0 \Rightarrow x_1 = 2 = x$$

$$x_2 - \underbrace{x_3}_{1} - 7\underbrace{x_4}_{-1} = 0 \Rightarrow x_2 = -6 = y$$

$$\vec{v}_4 = 2\vec{v}_1 - 6\vec{v}_2 + 1\vec{v}_3$$

also correct

Related to 6.2:

$$\begin{bmatrix} \vec{u}_1 & \vec{u}_2 & \vec{u}_3 & \vec{u}_4 \end{bmatrix} \xrightarrow[\text{op's}]{\text{row}} \begin{bmatrix} 1 & 0 & 0 & 9 \\ 0 & 0 & 1 & 3 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

RREF

$$\boxed{\vec{u}_4 = 9\vec{u}_1 + 3\vec{u}_3} \Leftarrow 9\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} + 3\begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} = \begin{bmatrix} 9 \\ 3 \\ 0 \end{bmatrix}$$

6.1 - related example:

$$\begin{bmatrix} \vec{v}_1 & \vec{v}_2 & \vec{v}_3 & \vec{v}_4 \end{bmatrix} \xrightarrow[\text{op's}]{\text{row}} \underbrace{\overset{\text{RREF}}{\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix}}}_{\text{all columns are pivot columns}}$$

$\vec{v}_1, \vec{v}_2, \vec{v}_3, \vec{v}_4$ are linearly independent $\Leftarrow$ all columns are pivot columns

---

$$\begin{bmatrix} 1 & 0 & 0 & 5 \\ 0 & 1 & 0 & 6 \\ 0 & 0 & 1 & 7 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \text{ row equiv to } \begin{bmatrix} \vec{c}_1 & \vec{c}_2 & \vec{c}_3 & \vec{c}_4 \end{bmatrix}$$

$\Rightarrow \quad 5\vec{c}_1 + 6\vec{c}_2 + 7\vec{c}_3 = \vec{c}_4 \Rightarrow$ linear dependence of $\vec{c}_1, \vec{c}_2, \vec{c}_3, \vec{c}_4$

5.2) If $\vec{x}$ solves $LS(A, \vec{b})$

& $A = \begin{bmatrix} \vec{a}_1 & \vec{a}_2 & \cdots & \vec{a}_n \end{bmatrix}$,

then $x_1 \vec{a}_1 + x_2 \vec{a}_2 + x_3 \vec{a}_3 + \cdots + x_n \vec{a}_n = \vec{b}$

E.g. $\begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix} = A$, $\begin{bmatrix} 7 \\ 19 \end{bmatrix} = \vec{b}$,

and $\vec{x} = \begin{bmatrix} 5 \\ 1 \end{bmatrix}$. $5\begin{bmatrix} 1 \\ 3 \end{bmatrix} + 1\begin{bmatrix} 2 \\ 4 \end{bmatrix} = \begin{bmatrix} 7 \\ 19 \end{bmatrix}$

$$\left.\begin{aligned} 1x_1 + 2x_2 &= 7 \\ 3x_1 + 4x_2 &= 19 \end{aligned}\right\} \Longleftrightarrow \left\{ x_1 \begin{bmatrix} 1 \\ 3 \end{bmatrix} + x_2 \begin{bmatrix} 2 \\ 4 \end{bmatrix} = \begin{bmatrix} 7 \\ 19 \end{bmatrix} \right.$$

5.4

$$N(A) = \left\{ x_3 \begin{bmatrix} -3 \\ 3 \\ 1 \\ 0 \\ 0 \\ 0 \end{bmatrix} + x_5 \begin{bmatrix} -4 \\ -7 \\ 0 \\ -9 \\ 1 \\ 0 \end{bmatrix} \;\middle|\; x_3, x_5 \in \mathbb{C} \right\} = \left\langle \left\{ \begin{bmatrix} -3 \\ 3 \\ 1 \\ 0 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} -4 \\ -7 \\ 0 \\ -9 \\ 1 \\ 0 \end{bmatrix} \right\} \right\rangle$$

5.6 No.

$$x_1 \begin{bmatrix} 1 \\ 1 \end{bmatrix} + x_2 \begin{bmatrix} 2 \\ 2 \end{bmatrix} + x_3 \begin{bmatrix} 3 \\ 3 \end{bmatrix} = \begin{bmatrix} 5 \\ 0 \end{bmatrix}$$

has no solution: $\left[\begin{array}{ccc|c} 1 & 2 & 3 & 5 \\ 1 & 2 & 3 & 0 \end{array}\right] \xrightarrow[-R_1+R_2]{} \left[\begin{array}{ccc|c} 1 & 2 & 3 & 5 \\ 0 & 0 & 0 & -5 \end{array}\right]$

5.5 $\begin{bmatrix} 1 & 2 & 3 & 4 \\ 2 & 3 & 4 & 5 \\ 4 & 5 & 6 & 7 \end{bmatrix} \xrightarrow[\text{op's}]{\text{row}} \begin{bmatrix} 1 & 0 & -1 & -2 \\ 0 & 1 & 2 & 3 \\ 0 & 0 & 0 & 0 \end{bmatrix}$

So, $\begin{bmatrix} 3 \\ 4 \\ 6 \end{bmatrix}, \begin{bmatrix} 4 \\ 5 \\ 7 \end{bmatrix} \in \left\langle \left\{ \begin{bmatrix} 1 \\ 2 \\ 4 \end{bmatrix}, \begin{bmatrix} 2 \\ 3 \\ 5 \end{bmatrix} \right\} \right\rangle$

So, $\left\langle \left\{ \begin{bmatrix} 1 \\ 2 \\ 4 \end{bmatrix}, \begin{bmatrix} 2 \\ 3 \\ 5 \end{bmatrix}, \begin{bmatrix} 3 \\ 4 \\ 6 \end{bmatrix}, \begin{bmatrix} 4 \\ 5 \\ 7 \end{bmatrix} \right\} \right\rangle$

$= \left\langle \left\{ \begin{bmatrix} 1 \\ 2 \\ 4 \end{bmatrix}, \begin{bmatrix} 2 \\ 3 \\ 5 \end{bmatrix} \right\} \right\rangle.$

We can solve this without computing the RREF too:

$\begin{bmatrix} 2 \\ 3 \\ 5 \end{bmatrix} - \begin{bmatrix} 1 \\ 2 \\ 4 \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} = \begin{bmatrix} 3 \\ 4 \\ 6 \end{bmatrix} - \begin{bmatrix} 2 \\ 3 \\ 5 \end{bmatrix} = \begin{bmatrix} 4 \\ 5 \\ 7 \end{bmatrix} - \begin{bmatrix} 3 \\ 4 \\ 6 \end{bmatrix}$

$$\begin{bmatrix} 1 & 2 & 3 & 4 \\ 2 & 3 & 4 & 5 \\ 4 & 5 & 6 & 7 \end{bmatrix} = \begin{bmatrix} \vec{a} & \vec{b} & \vec{c} & \vec{d} \end{bmatrix} \text{ for short.}$$

$$\vec{b} - \vec{a} = \vec{c} - \vec{b} = \vec{d} - \vec{c}$$

$$\Rightarrow \vec{c} = 2\vec{b} - \vec{a} \qquad \vec{d} = 2\vec{c} - \vec{b}$$

$$\Downarrow \qquad\qquad \Downarrow$$

$$\vec{c} \in \langle \{\vec{a}, \vec{b}\} \rangle \;\&\; \vec{d} \in \langle \{\vec{a}, \vec{b}, \vec{c}\} \rangle$$

$$\Downarrow$$

$$\langle \{\vec{a}, \vec{b}, \vec{c}, \vec{d}\} \rangle = \langle \{\vec{a}, \vec{b}\} \rangle$$

E.g. $4\vec{a} + 7\vec{b} - 3\vec{c} + 2\vec{d}$

$= 4\vec{a} + 7\vec{b} - 3\vec{c} + 2(2\vec{c} - \vec{b})$

$= 4\vec{a} + 7\vec{b} - 3(2\vec{b} - \vec{a}) + 2(2(2\vec{b} - a) - \vec{b})$

6.3 $0(\text{col. }1) + 5(\text{col. }2) + 1(\text{col. }3)$

$= \vec{0} \Rightarrow$ dependent

6.4 all pivot colums $\Rightarrow$ independent