Practice problems (Day 1)

(1)
$$5x + 3y = 1$$
$$10x + 4y = 1$$

1a) Find the solution set.

1b) Change just one of the six constants in the system so as to make it inconsistent.

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(2)
$$7x - y = 3$$
$$14x - 2y = 6$$
$$-x + \frac{1}{7}y = -\frac{3}{7}$$

Find the solution set and plot it.

Want more? Try section SSLE exercises.

Day 1

1a. $\left[\begin{array}{cc|c} 5 & 3 & 1 \\ 10 & 4 & 1 \end{array}\right] \longrightarrow \begin{array}{c} \\ -2R_1+R_2 \end{array} \left[\begin{array}{cc|c} 5 & 3 & 1 \\ 0 & -2 & -1 \end{array}\right]$.

$\longrightarrow \begin{array}{c} \frac{1}{5}R_1 \\ \\ \end{array} \left[\begin{array}{cc|c} 1 & .6 & .2 \\ 0 & -2 & -1 \end{array}\right] \longrightarrow \begin{array}{c} \\ -\frac{1}{2}R_2 \end{array} \left[\begin{array}{cc|c} 1 & .6 & .2 \\ 0 & 1 & .5 \end{array}\right]$

$\longrightarrow -.6R_2+R_1 \left[\begin{array}{cc|c} 1 & 0 & -.1 \\ 0 & 1 & .5 \end{array}\right]$ $\boxed{\begin{array}{l} x = -1/10 \\ y = 1/2 \end{array}}$

1b. $\underbrace{\begin{array}{r} 5x + 3y = 1 \\ 10x + \underline{6}y = 1 \end{array}}_{\text{inconsistent system}}$

The second equation's left side is twice the 1st eqn's left side. But the right sides are equal.

2. Row 2 is twice Row 1. Row 3 is $-\frac{1}{7}$ of Row 1.
So, the solution set is given by Row 1:
$\{(x,y) \mid 7x - y = 3\} = \{(x, 7x-3) \mid x \in \mathbb{C}\}$.
See plot 3 pages down...

(1) For each matrix below, determine if it is RREF. Also: Whether RREF or not, circle the pivots. (Day 2)

$$A = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 5 \\ 0 & 0 & 1 & 7 \end{bmatrix} \quad B = \begin{bmatrix} 5 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

$$C = \begin{bmatrix} 1 & 2 & 3 \\ 0 & 1 & 4 \\ 0 & 0 & 1 \end{bmatrix} \quad D = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 1 & 0 & 0 & 1 \end{bmatrix}$$

$$E = \begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix} \quad F = \begin{bmatrix} 1 & 4 & 0 & 1 & 0 & 6 \\ 0 & 0 & 1 & 2 & 0 & 7 \\ 0 & 0 & 0 & 0 & 1 & 5 \end{bmatrix}$$

$$G = \begin{bmatrix} 0 & 1 & 2 & 3 & 0 & 0 & 4 & 0 & 6 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 5 & 0 \end{bmatrix}$$

Only F & G are RREF.

(2) Interpret matrix F from (1) as a (linear) system of equations.

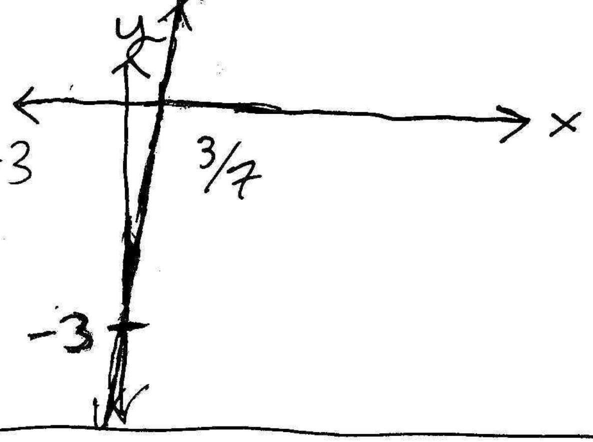
Day 2

(3) Represent
$$\begin{aligned} x+2y &= 3 \\ 3x+4y &= 5 \end{aligned}$$
with an augmented matrix and find a sequence of three row operations that reach RREF.

(4) Can a matrix have more pivot rows than pivot columns? Justify your answer.

(5) How can a computer help you check whether two systems of linear equations are equivalent?

Plot of real solutions:



A line with y-intercept -3 & x-intercept $+3/7$.

Day 1

A is not RREF: zero row above pivot row. (1)

Day 2

B is not RREF: top row nonzero & non-pivot.

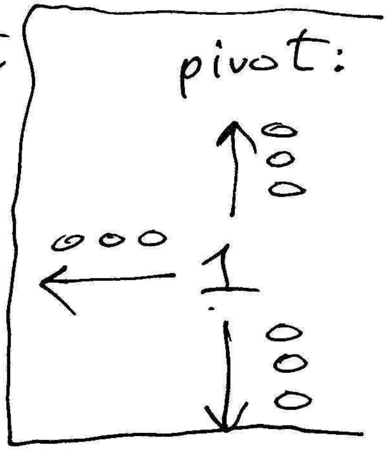
C is not RREF: 2nd row nonzero & non-pivot.

D is not RREF: leftmost pivot not highest pivot

E is not RREF: top row nonzero & non-pivot

F is RREF. G is RREF.

RREF: pivot rows above zero rows, and no other kind of row; pivots descend to right.



(2)

$$x_1 + 4x_2 \quad + x_4 = 6$$
$$x_3 + 2x_3 = 7$$
$$x_5 = 5$$

Day 2

(3) $x+2y=3$
$3x+4y=5$

$$\longrightarrow \left[\begin{array}{cc|c} 1 & 2 & 3 \\ 3 & 4 & 5 \end{array}\right] \xrightarrow{-3R_1+R_2} \left[\begin{array}{cc|c} 1 & 2 & 3 \\ 0 & -2 & -4 \end{array}\right]$$

$$\xrightarrow{R_2+R_1} \left[\begin{array}{cc|c} 1 & 0 & -1 \\ 0 & -2 & -4 \end{array}\right] \xrightarrow{-\frac{1}{2}R_2} \left[\begin{array}{cc|c} 1 & 0 & -1 \\ 0 & 1 & 2 \end{array}\right]$$

(4) No. If it did, two pivot rows would contain pivots in the same column. But the higher pivot is a 1 with 0's only below. So, we can't have another pivot beneath it.

(5) Expressing each system as an augmented matrix, we ask the computer to compute the RREF of each. The systems are equivalent if and only if the RREF's match, except possibly one having more zero rows than the other.

Day 3

$$\begin{array}{c} \begin{array}{cccccc} P & F & P & P & F & P \end{array} \\ A = \begin{bmatrix} 1 & 1 & 0 & 0 & -1 & 0 \\ 0 & 0 & 1 & 0 & 2 & 0 \\ 0 & 0 & 0 & 1 & 3 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix}; \end{array} \quad \vec{b} = \begin{bmatrix} 4 \\ 5 \\ 6 \\ 7 \end{bmatrix}$$

1. Find the solution set of $LS(A, \vec{b})$.
2. Give 3 examples of solutions to $LS(A, \vec{b})$
3. Give an example of a 3×3 (coefficient) matrix $\begin{bmatrix} ? & ? & ? \\ ? & ? & ? \\ ? & ? & ? \end{bmatrix}$ that is in RREF and has two free variables in its null space.
4. Find an example of a 5-row RREF matrix with null space $\left\{ \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \right\}$.
5. Find the solution set for augmented...

Day 3

(1)
$$\begin{aligned}
x_1 + x_2 \qquad\qquad - x_5 &= 4\\
x_3 \quad + 2x_5 &= 5\\
x_4 + 3x_5 &= 6\\
x_6 &= 7
\end{aligned}
\Rightarrow$$

Solution set:

$$\left\{ \begin{bmatrix} 4 - x_2 + x_5 \\ x_2 \\ 5 - 2x_5 \\ 6 - 3x_5 \\ x_5 \\ 7 \end{bmatrix} \;\middle|\; x_2, x_5 \in \mathbb{C} \right\}$$

(2)
$$\begin{bmatrix} 4 \\ 0 \\ 5 \\ 6 \\ 0 \\ 7 \end{bmatrix}, \begin{bmatrix} 3 \\ 1 \\ 5 \\ 6 \\ 0 \\ 7 \end{bmatrix}, \begin{bmatrix} 5 \\ 0 \\ 3 \\ 3 \\ 1 \\ 7 \end{bmatrix}$$

← Used $(x_2, x_5) = (0,0)$, $(x_2, x_5) = (1,0)$, & $(x_2, x_5) = (0,1)$.

(3)
$$\begin{array}{ccc} & & \\ \multicolumn{3}{c}{\begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}} \\ F & P & F \end{array}$$

(The null space corresponds to
$$\left[\begin{array}{ccc|c} 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right].)$$

(4)
$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

5 equations

3 variables in <u>coefficient</u> matrix.

...matrix $\begin{bmatrix} 1 & 1 & 0 & 0 & -1 & 0 & 0 \\ 0 & 0 & 1 & 0 & 2 & 0 & 0 \\ 0 & 0 & 0 & 1 & 3 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}$.

Day 3

6) If a linear system has two solutions, then it also has infinitely many solutions.
(Fill in the blank.)

7) Is your example for 3 unique, or is there another example? Not unique.

8) Is your example for 4 unique, or is there another example? Unique.

⑤ $\left.\begin{aligned} x_1 + x_2 \qquad\qquad - x_5 &= 0 \\ x_3 \qquad + 2x_5 &= 0 \\ x_4 + 3x_5 &= 0 \\ x_6 &= 0 \\ 0 &= 1 \end{aligned}\right\}$

Day 3

The solution set is the empty set ϕ.

~~6~~ ⑦ Any 3×3 matrix with a pivot row ~~about~~ above two zero rows is an example. (E.g., $\begin{bmatrix} 1 & 2 & 3 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$)

⑧ Because there are only 3 columns, there is only one way to have 3 pivots descending to the right, with the zero rows beneath the pivot rows. We need all the columns to be pivot columns to ensure a null space of $\{\vec{0}\}$.

(Day 4)

① If A is a 4×4 matrix and $\begin{bmatrix} 5 \\ 0 \\ 0 \\ 1 \end{bmatrix} \in N(A)$, what can you say about $LS\left(A, \begin{bmatrix} 2 \\ 3 \\ 4 \\ 5 \end{bmatrix}\right)$?

② Assume A is a 4×7 matrix and A is row-equivalent to

$$R = \begin{bmatrix} 1 & 4 & 0 & -1 & 0 & 0 & 5 \\ 0 & 0 & 1 & 2 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 3 \\ 0 & 0 & 0 & 0 & 0 & 1 & -3 \end{bmatrix}$$

Fill in the blank vectors:

Day 4

$$N(A) = \left\{ x_2 \begin{bmatrix} -4 \\ 1 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix} + x_4 \begin{bmatrix} 1 \\ 0 \\ -2 \\ 1 \\ 0 \\ 0 \\ 0 \end{bmatrix} + x_7 \begin{bmatrix} -5 \\ 0 \\ 0 \\ 0 \\ -3 \\ 3 \\ 1 \end{bmatrix} \;\middle|\; x_2, x_4, x_7 \in \mathbb{C} \right\}$$

(Each column vector should have 7 rows.)

(3) $20 \begin{bmatrix} 4 \\ -3 \end{bmatrix} - 7 \begin{bmatrix} 1 \\ 2 \end{bmatrix} = ?$

Day 5

(1) Assuming $\begin{bmatrix} 0 \\ 1 \\ 2 \end{bmatrix} \in N(A)$ and

$\begin{bmatrix} 5 \\ 1 \\ 7 \end{bmatrix}$ is a solution to $LS(A, \vec{b})$,

find three other solutions to $LS(A, \vec{b})$.

(2) If $LS(A, \vec{b}) \neq \phi$, then $\vec{b}$ is

a linear combination of columns of A.

(3) If $LS(A, \vec{b}) = \phi$, then $\vec{b}$ is

NOT a linear combination columns of A.

For (2) & (3), fill in the blank.

Day 4

① If $\vec{x}$ solves $LS\left(A, \begin{bmatrix} 2 \\ 3 \\ 4 \\ 5 \end{bmatrix}\right)$, then so does $\vec{x} + c\begin{bmatrix} 5 \\ 0 \\ 0 \\ 1 \end{bmatrix}$ for all $c \in \mathbb{C}$. Therefore, $LS\left(A, \begin{bmatrix} 2 \\ 3 \\ 4 \\ 5 \end{bmatrix}\right)$ either has no solutions or infinitely many.

③ $\begin{bmatrix} 80 \\ -60 \end{bmatrix} + \begin{bmatrix} -7 \\ -14 \end{bmatrix} = \begin{bmatrix} 73 \\ -74 \end{bmatrix}$

Day 5

① $\begin{bmatrix} 5 \\ 1 \\ 7 \end{bmatrix} + c\begin{bmatrix} 0 \\ 1 \\ 2 \end{bmatrix}$ for three nonzero $c \in \mathbb{C}$:

$\begin{bmatrix} 5 \\ 2 \\ 9 \end{bmatrix}, \begin{bmatrix} 5 \\ 3 \\ 11 \end{bmatrix}, \begin{bmatrix} 5 \\ 0 \\ 5 \end{bmatrix}$ for $c = 1, 2, -1$.

Day 5

(4) Given $A=\begin{bmatrix} 1 & 0 & 3 & 0 & 4 & 0 \\ 0 & 1 & -3 & 0 & 7 & 0 \\ 0 & 0 & 0 & 1 & 9 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix}$, express $N(A)$ as a span of two vectors.

starting steps $\rightarrow$

$$x_1+3x_3+4x_5=0$$
$$x_2-3x_3+7x_5=0$$
$$x_4+9x_5=0$$
$$x_6=0$$

(5) Express $\left\langle\left\{\begin{bmatrix}1\\2\\4\end{bmatrix},\begin{bmatrix}2\\3\\5\end{bmatrix},\begin{bmatrix}3\\4\\6\end{bmatrix},\begin{bmatrix}4\\5\\7\end{bmatrix}\right\}\right\rangle$ as a span of two vectors.

(6) Is $\begin{bmatrix}5\\0\end{bmatrix}\in\left\langle\left\{\begin{bmatrix}1\\1\end{bmatrix},\begin{bmatrix}2\\2\end{bmatrix},\begin{bmatrix}3\\3\end{bmatrix}\right\}\right\rangle$? Explain why or why not.

<u>No.</u> $\left\langle\left\{\begin{bmatrix}1\\1\end{bmatrix},\begin{bmatrix}2\\2\end{bmatrix},\begin{bmatrix}3\\3\end{bmatrix}\right\}\right\rangle=\left\{\begin{bmatrix}x\\x\end{bmatrix}\middle|\, x\in\mathbb{C}\right\}$ & $5\neq 0$.

Day 5

(4) $$N(A)=\left\{\begin{bmatrix}-3x_3-4x_5\\3x_3-7x_5\\x_3\\-9x_5\\x_5\\0\end{bmatrix}\;\middle|\; x_2, x_5\in\mathbb{C}\right\}=\left\langle\left\{\begin{bmatrix}-3\\3\\1\\0\\0\\0\end{bmatrix},\begin{bmatrix}-4\\-7\\0\\-9\\1\\0\end{bmatrix}\right\}\right\rangle$$

span notation $\left\langle\left\{\begin{bmatrix}\#\\ \vdots\end{bmatrix},\dots,\begin{bmatrix}\#\\ \vdots\end{bmatrix}\right\}\right\rangle$

(5) $$\begin{bmatrix}2\\3\\5\end{bmatrix}-\begin{bmatrix}1\\2\\4\end{bmatrix}=\begin{bmatrix}3\\4\\6\end{bmatrix}-\begin{bmatrix}2\\3\\5\end{bmatrix}=\begin{bmatrix}4\\5\\7\end{bmatrix}-\begin{bmatrix}3\\4\\6\end{bmatrix}\left(=\begin{bmatrix}1\\1\\1\end{bmatrix}\right).$$

So, $$\begin{bmatrix}3\\4\\6\end{bmatrix}=2\begin{bmatrix}2\\3\\5\end{bmatrix}-\begin{bmatrix}1\\2\\4\end{bmatrix} \;\&\; \begin{bmatrix}4\\5\\7\end{bmatrix}=2\begin{bmatrix}3\\4\\6\end{bmatrix}-\begin{bmatrix}2\\3\\5\end{bmatrix}.$$

So, the last vector is in the span of the first three and the 3rd vector is in the span of the first two. So, the first two have the same span as all four:

$$\left\langle\left\{\begin{bmatrix}1\\2\\4\end{bmatrix},\begin{bmatrix}2\\3\\5\end{bmatrix},\begin{bmatrix}3\\4\\6\end{bmatrix},\begin{bmatrix}4\\5\\7\end{bmatrix}\right\}\right\rangle=\left\langle\left\{\begin{bmatrix}1\\2\\4\end{bmatrix},\begin{bmatrix}2\\3\\5\end{bmatrix}\right\}\right\rangle.$$

Day 6

$$\vec{v}_1 = \begin{bmatrix} 1 \\ 2 \\ 3 \\ 2 \end{bmatrix}, \quad \vec{v}_2 = \begin{bmatrix} 3 \\ 1 \\ 0 \\ 1 \end{bmatrix}, \quad \vec{v}_3 = \begin{bmatrix} 0 \\ 5 \\ 9 \\ 5 \end{bmatrix}$$

The RREF of $A = \begin{bmatrix} 1 & 3 & 0 \\ 2 & 1 & 5 \\ 3 & 0 & 9 \\ 2 & 1 & 5 \end{bmatrix}$

is $R = \begin{bmatrix} 1 & 0 & 3 \\ 0 & 1 & -1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$.

1. Are $\vec{v}_1, \vec{v}_2, \vec{v}_3$ linearly dependent, or linearly independent?

2. $\vec{v}_4 = \begin{bmatrix} -16 \\ 3 \\ 15 \\ 3 \end{bmatrix}$ and the RREF...

...of $\begin{bmatrix} 1 & 3 & 0 & -16 \\ 2 & 1 & 5 & 3 \\ 3 & 0 & 9 & 15 \\ 2 & 1 & 5 & 3 \end{bmatrix}$ is $\begin{bmatrix} 1 & 0 & 3 & 5 \\ 0 & 1 & -1 & -7 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$.

Use this information to find $x, y, z \in \mathbb{C}$ such that $\vec{v}_4 = x\vec{v}_1 + y\vec{v}_2 + z\vec{v}_3$.

Day 6

(3) If $N(B) = \left\langle \left\{ \begin{bmatrix} 0 \\ 5 \\ 1 \end{bmatrix} \right\} \right\rangle$, then the columns of B are linearly <u>dependent</u>.

(4) If the RREF of matrix C has 8 rows, 5 columns, and 5 pivots, then the columns of C are linearly <u>independent</u>.

Day 6

① Column 3 of R tells us that

$\vec{v}_3 = 3\vec{v}_1 - \vec{v}_2 \in \langle\{\vec{v}_1, \vec{v}_2\}\rangle$: $3\begin{bmatrix}1\\0\\0\\0\end{bmatrix} - \begin{bmatrix}0\\1\\0\\0\end{bmatrix} = \begin{bmatrix}3\\-1\\0\\0\end{bmatrix}$.

So, $\vec{v}_1, \vec{v}_2, \vec{v}_3$ are linearly dependendent.

② $\vec{v}_4 = \underline{5}\,\vec{v}_1 \underline{-7}\,\vec{v}_2 + \underline{0}\,\vec{v}_3$: $5\begin{bmatrix}1\\0\\0\\0\end{bmatrix} + (-7)\begin{bmatrix}0\\1\\0\\0\end{bmatrix} + 0\begin{bmatrix}3\\-1\\0\\0\end{bmatrix} = \begin{bmatrix}5\\-7\\0\\0\end{bmatrix}$

③ $\begin{bmatrix}0\\5\\1\end{bmatrix} \in N(B) \iff 5(\text{column } 2) + (\text{column } 3) = \vec{0}$.

④ RREF $R = \begin{bmatrix}1&&&&\\&1&&&\\&&1&&\\&&&1&\\&&&&1\\&&&&\\&&&&\end{bmatrix}$ has linearly independent columns.

So, C does too.

Comment: If matrix A has RREF $B = \begin{bmatrix}| & | & & | \\ \vec{b}_1 & \vec{b}_2 & \cdots & \vec{b}_n \\ | & | & & |\end{bmatrix}$

and $c_1\vec{b}_1 + \cdots + c_n\vec{b}_n = 0$, then the same is true of A's columns.