(1) Which of the following are subspaces of $\mathbb{C}^3$?

$\boxed{\text{Day 14}}$

$P = \left\{ \begin{bmatrix} x \\ y \\ z \end{bmatrix} \middle| \; x + y + z = 0 \right\}$, $Q = \left\{ \begin{bmatrix} x \\ y \\ z \end{bmatrix} \middle| \; x + y = 0 \right\}$,

$R = R\left(\begin{bmatrix} 1 & 2 \\ 3 & 4 \\ 5 & 6 \end{bmatrix} \right)$, $S = C\left(\begin{bmatrix} 1 & 2 \\ 3 & 4 \\ 5 & 6 \end{bmatrix} \right)$,

$T = \left\{ \begin{bmatrix} x \\ y \\ z \end{bmatrix} \middle| \; x + y + z = 1 \right\}$, $U = \left\{ \begin{bmatrix} x \\ y \\ z \end{bmatrix} \middle| \; xy = 0 \right\}$,

$V = \left\{ \begin{bmatrix} x \\ y \\ z \end{bmatrix} \middle| \; x + y = 0 \text{ or } y + z = 0 \right\}$, $W = N\left(\begin{bmatrix} 1 & 2 \\ 3 & 4 \\ 5 & 6 \end{bmatrix} \right)$,

$X = L\left(\begin{bmatrix} 1 & 2 \\ 3 & 4 \\ 5 & 6 \end{bmatrix} \right)$, $Y = \left\{ \begin{bmatrix} x \\ y \\ z \end{bmatrix} \middle| \; x + y = 0 \text{ and } y + z = 0 \right\}$.

(14

(2) Which of the following are subspaces of P_5? (Notation: $p' = dp/dx$; $p(a) = $ (p at $x=a$))

$A = \langle\{5-x, x^2, x^5\}\rangle$, $B = \{p \in P_5 \mid p(3) = 0\}$,

$C = \{p \in P_5 \mid p' = 0\}$, $D = \{p \in P_5 \mid p'''(3) = 0\}$,

$E = \{p \in P_5 \mid 2p(1) = p'(0) + p''(-1)\}$,

$F = \{p \in P_5 \mid \int_0^5 p(t)\,dt = \int_1^2 p(t)\,dt\}$,

$G = \{p \in P_5 \mid \int_0^5 p(t)\,dt = 3\}$, $H = P_4$,

$I = \{p \in P_5 \mid \frac{d}{dx}(x^2 p(x)) = 3x\,p(x)\}$,

$J = \{p \in P_5 \mid \frac{d}{dx}(x^2 p(x)) = 3x\}$, $K = \{0\}$.

③ Which of the following are subspaces of M_{33}?

$U = \{A \in M_{33} \mid A \text{ is unitary}\}$,

$H = \{A \in M_{33} \mid A \text{ is self-adjoint}\}$,

$N = \{A \in M_{33} \mid A\begin{bmatrix}1\\2\\3\end{bmatrix} = \begin{bmatrix}0\\0\\0\end{bmatrix}\}$,

$E = \{A \in M_{33} \mid A\begin{bmatrix}1\\2\\3\end{bmatrix} = A\begin{bmatrix}4\\5\\6\end{bmatrix}\}$,

$R = \{A \in M_{33} \mid \begin{bmatrix}7&3&5\\8&2&1\end{bmatrix}A = \begin{bmatrix}0&0&0\\0&0&0\end{bmatrix}\}$,

$V = \{A \in M_{33} \mid A \text{ is invertible}\}$,

$S = \{A \in M_{33} \mid A \text{ is singular}\}$,

$T = \{A \in M_{33} \mid A = A^t\}$.

Day 14

① P, Q, R, S, W, X, Y are subspaces of vector spaces (and so are vector spaces). But $R \subset \mathbb{C}^2$, not $\mathbb{C}^3$. $W \subset \mathbb{C}^2$ too. So, $\boxed{P, Q, S, X, Y}$ are the subspaces of $\mathbb{C}^3$.

T is not a subspace of $\mathbb{C}^3$ because $\vec{0} \notin T$.

U is not because $\begin{bmatrix}1\\0\\0\end{bmatrix}, \begin{bmatrix}0\\1\\0\end{bmatrix} \in U$ but $\begin{bmatrix}1\\0\\0\end{bmatrix} + \begin{bmatrix}0\\1\\0\end{bmatrix} \notin U$.

V is not because $\begin{bmatrix}0\\0\\1\end{bmatrix}, \begin{bmatrix}1\\0\\0\end{bmatrix} \in V$ but $\begin{bmatrix}0\\0\\1\end{bmatrix} + \begin{bmatrix}1\\0\\0\end{bmatrix} \notin V$.

Note that $P = N([1 \ 1 \ 1])$, $Q = N([1 \ 1 \ 0])$, and $Y = N\left(\begin{bmatrix}1 & 1 & 0\\0 & 1 & 1\end{bmatrix}\right)$.

(In fact, every subspace of $\mathbb{C}^n$ equals some $N(A)$. But finding that A is not always easy.)

② $\boxed{A, B, C, D, E, F, H, I, K}$ are subspaces of P_5. G and J are not because $0 \notin G$ and $0 \notin J$. All the other given subsets of P_5 contain 0, are additively closed (if f & g are in, then $f+g$ is in), and homogeneous (if $c \in \mathbb{C}$ and f is in, cf is in).

Day 14

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③ $\begin{bmatrix} 0&0&0 \\ 0&0&0 \\ 0&0&0 \end{bmatrix} \notin U, \cancel{E}, V$. $I_3 \in H$, but $iI_3 \notin H$.

$\begin{bmatrix} 1&0&0 \\ 0&1&0 \\ 0&0&0 \end{bmatrix}, \begin{bmatrix} 0&0&0 \\ 0&0&0 \\ 0&0&1 \end{bmatrix} \in S$ but $\begin{bmatrix} 1&0&0 \\ 0&1&0 \\ 0&0&0 \end{bmatrix} + \begin{bmatrix} 0&0&0 \\ 0&0&0 \\ 0&0&1 \end{bmatrix} \notin S$.

E, N, R, T are subspaces of M_{33}: for each of these subsets, $\begin{bmatrix} 0&0&0 \\ 0&0&0 \\ 0&0&0 \end{bmatrix}$ is in, if A & B are in, then $A+B$ is in, and if A is in & $\alpha \in \mathbb{C}$, then αA is in.

Day 15

① Find a basis for $\langle x^3+1,\ 1+x^2,\ x^2(1-x)\rangle = V$, subspace of P_3.

② Find a basis for $\{p \in P_2 \mid \int_0^1 p(x^2)\,dx = 0\}$.

③ Find a basis for $\{A \in M_{22} \mid A = A^t\}$.

④ Find a basis for $\{A \in M_{22} \mid A = -A^t\}$.

⑤ Find a basis for $\langle \cos(x+\frac{\pi}{4}),\ \sin(x-\frac{\pi}{3}),\ \cos(x+\frac{\pi}{3}),\ \sin(\frac{\pi}{6}-x)\rangle$, subspace of $C^{\mathbb{R}}$.

Hint: $\cos(\alpha+\beta) = \cos\alpha\cos\beta - \sin\alpha\sin\beta$
$\sin(\alpha+\beta) = \cos\alpha\sin\beta + \sin\alpha\cos\beta$

Day 15

(1) $x^2(1-x) = x^2 - x^3 = (1+x^2) - (x^3+1)$. So, $\langle\{x^3+1, 1+x^2\}\rangle = V$. Since x^3+1 is not $\alpha(1+x^2)$ for any $\alpha \in \mathbb{C}$, $\{x^3+1, 1+x^2\}$ is a basis for V, for this set is linearly independent & spans V.

(2) $\int_0^1 \left(a(x^2)^2 + bx^2 + c\right)dx = \frac{a}{5} + \frac{b}{3} + c = 0 \Longleftrightarrow c = -\frac{a}{5} - \frac{b}{3}$

a&b are free variables!

So, using $(a,b) = (1,0)$ & $(0,1)$, we get a basis:

$\left\{x^2 - \frac{1}{5},\ x - \frac{1}{3}\right\}$.

(3) $\left\{\begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}, \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}\right\}$ is a basis for

$$\left\{\begin{bmatrix} a & c \\ c & b \end{bmatrix} \middle|\ a, b, c \in \mathbb{C}\right\}.$$

(4) $\left\{\begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}\right\}$ is a basis for $\left\{\begin{bmatrix} 0 & -a \\ a & 0 \end{bmatrix} \middle|\ a \in \mathbb{C}\right\}$.

(5) $f = \cos\left(x + \frac{\pi}{4}\right) = \cos(x)\cos\left(\frac{\pi}{4}\right) - \sin(x)\sin\left(\frac{\pi}{4}\right)$

Day 15

$g = \sin\left(x - \frac{\pi}{3}\right) = \cos(x)\sin\left(-\frac{\pi}{3}\right) + \sin(x)\cos\left(-\frac{\pi}{3}\right)$

$h = \cos\left(x + \frac{\pi}{3}\right) = \cos(x)\cos\left(\frac{\pi}{3}\right) - \sin(x)\sin\left(\frac{\pi}{3}\right)$

$k = \sin\left(\frac{\pi}{6} - x\right) = \cos\left(\frac{\pi}{6}\right)\underbrace{\sin(-x)}_{-\sin(x)} + \sin\left(\frac{\pi}{6}\right)\underbrace{\cos(-x)}_{\cos x}$

Thus, $\langle\{f, g, h, k\}\rangle \subset \langle\{\cos x, \sin x\}\rangle$.

Conversely, $\langle\{\cos x, \sin x\}\rangle \subset \langle\{g, h\}\rangle \subset \langle\{f, g, h, k\}\rangle$

because $\begin{bmatrix} h \\ g \end{bmatrix} = \underbrace{\begin{bmatrix} \cos\left(\frac{\pi}{3}\right) & -\sin\left(\frac{\pi}{3}\right) \\ -\sin\left(\frac{\pi}{3}\right) & \cos\left(\frac{\pi}{3}\right) \end{bmatrix}}_{A} \begin{bmatrix} \cos(x) \\ \sin(x) \end{bmatrix}$

$\Rightarrow \begin{bmatrix} \cos(x) \\ \sin(x) \end{bmatrix} = A^{-1}\begin{bmatrix} h \\ g \end{bmatrix}$ because A^{-1} exists because

$\cos\left(\frac{\pi}{3}\right)\cos\left(\frac{\pi}{3}\right) - \sin\left(\frac{\pi}{3}\right)\sin\left(\frac{\pi}{3}\right) = \cos\left(\frac{\pi}{3} + \frac{\pi}{3}\right) \neq 0.$

Day 15

So, $\boxed{\{\cos x, \sin x\}}$ has the span we want. It is also a basis for $\langle\{f, g, h, k\}\rangle$ because $\cos x \neq \alpha \sin x$ for all $\alpha \in \mathbb{C}$. (By this I mean that if $\alpha \in \mathbb{C}$, then there is some x such that $\cos x \neq \alpha \sin x$. For example, $x = 0$: $\cos 0 = 1 \neq 0 = \alpha \cdot 0 = \alpha \sin 0$.)

Note: I didn't ask for a proof. It would be fine to just answer "$\{\cos x, \sin x\}$" or "$\cos x, \sin x$".

Alternative bases include any two of f, g, h. (However, $\{h, k\}$ is not a basis because $h = k$ because $\sin\frac{\pi}{3} = \cos\frac{\pi}{6}$ & $\sin\frac{\pi}{6} = \cos\frac{\pi}{3}$.)

1. Find the coordinates for $\begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}$ with respect to the basis

Day 16

$$\begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}, \begin{bmatrix} 0 & 1 \\ 1 & 1 \end{bmatrix}, \begin{bmatrix} 0 & 0 \\ 1 & 1 \end{bmatrix}, \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}.$$

2. Find the coordinates of $\begin{bmatrix} 5 \\ 6 \end{bmatrix}$ with respect to the basis $\begin{bmatrix} 2 \\ 1 \end{bmatrix}, \begin{bmatrix} 5 \\ -1 \end{bmatrix}$.

3. If p has coordinates $\begin{bmatrix} -1 \\ 0 \\ 8 \end{bmatrix}$ with respect to basis $(x+3)^2$, $x+5$, x^2, then what are the coordinates of p with respect to basis $1, x, x^2$?

Day 16

(4) If $\vec{v}$ has coordinates $\begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}$ with respect to basis $\begin{bmatrix} 4 \\ 0 \\ 4 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 7 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 0 \\ 3 \end{bmatrix}$ of subspace $W = \{ \vec{x} \in \mathbb{C}^4 \mid x_1 = x_3 \}$, then what are the coordinates of $\vec{v}$ with respect to basis $\begin{bmatrix} 2 \\ 0 \\ 2 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ -1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 0 \\ 10 \end{bmatrix}$ of W?

Day 16

① $$\begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix} = x_1 \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix} + x_2 \begin{bmatrix} 0 & 1 \\ 1 & 1 \end{bmatrix} \cancel{+x_3}$$

$$+ x_3 \begin{bmatrix} 0 & 0 \\ 1 & 1 \end{bmatrix} + x_4 \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} x_1 & x_1 + x_2 \\ x_1 + x_2 + x_3 & x_1 + x_2 + x_3 + x_4 \end{bmatrix}$$

has (unique) solution $\vec{x} = \begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \end{bmatrix}$.

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② $$\begin{bmatrix} 5 \\ 6 \end{bmatrix} = x_1 \begin{bmatrix} 2 \\ 1 \end{bmatrix} + x_2 \begin{bmatrix} 5 \\ -1 \end{bmatrix} = \begin{bmatrix} 2 & 5 \\ 1 & -1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$$

$$\Longleftrightarrow \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 2 & 5 \\ 1 & -1 \end{bmatrix}^{-1} \begin{bmatrix} 5 \\ 6 \end{bmatrix} = \frac{1}{2(-1) - 5(1)} \begin{bmatrix} -1 & -5 \\ -1 & 2 \end{bmatrix} \begin{bmatrix} 5 \\ 6 \end{bmatrix}$$

$$= -\frac{1}{7} \begin{bmatrix} -35 \\ 7 \end{bmatrix} = \begin{bmatrix} 5 \\ -1 \end{bmatrix}$$

③ $p = (-1)(x+3)^2 + 0(x+5) + 8x^2$

Day 16

$= -(x^2+6x+9) + 8x^2 = -9 - 6x + 7x^2$

So, p's coordinates are $\begin{bmatrix} -9 \\ -6 \\ 7 \end{bmatrix}$ for basis $1, x, x^2$.

④ $\vec{v} = 1\begin{bmatrix} 4 \\ 0 \\ 4 \\ 0 \end{bmatrix} + 2\begin{bmatrix} 0 \\ 7 \\ 0 \\ 0 \end{bmatrix} + 3\begin{bmatrix} 0 \\ 0 \\ 0 \\ 3 \end{bmatrix}$

$= 2\begin{bmatrix} 2 \\ 0 \\ 2 \\ 0 \end{bmatrix} - 14\begin{bmatrix} 0 \\ -1 \\ 0 \\ 0 \end{bmatrix} + \frac{9}{10}\begin{bmatrix} 0 \\ 0 \\ 0 \\ 10 \end{bmatrix}.$

So, $\vec{v}$'s coordinates are $\begin{bmatrix} 2 \\ -14 \\ 9/10 \end{bmatrix}$

for the requested basis.

Day 17

① To "rescale" by a factor of 3 in the x-direction & a factor of 4 in the y-direction, in 2D space ($\mathbb{C}^2$), use $A = \begin{bmatrix} 3 & 0 \\ 0 & 4 \end{bmatrix}$: $\begin{bmatrix} x \\ y \end{bmatrix}$ is transformed to $A\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 3x \\ 4y \end{bmatrix}$.

Find the matrix B that rescales by a factor of 3 in the $\vec{u}$-direction and 4 in the $\vec{v}$-direction, where $\vec{u}$ & $\vec{v}$ are $\vec{e}_1$ & $\vec{e}_2$ rotated $30°$ counterclockwise.

Day 17

(2) Suppose that D is a 6×7 matrix and

$$D \xrightarrow{\text{RREF}} \begin{bmatrix} 1 & -1 & 0 & 2 & 1 & 0 & 0 \\ 0 & 0 & 1 & 3 & -4 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

$\dim(C(D)) = \_\_\_\_$.

$\dim(N(D)) = \_\_\_\_$.

$\dim(R(D)) = \_\_\_\_$.

$\dim(L(D)) = \_\_\_\_$.

(3) True/False: If E is a square matrix, then $\dim(N(E)) = \dim(L(E))$. If true, explain why. If false, give a counterexample.

① Let $U = \begin{bmatrix} \cos(30^\circ) & -\sin(30^\circ) \\ \sin(30^\circ) & \cos(30^\circ) \end{bmatrix} = \begin{bmatrix} \underbrace{\begin{matrix} \sqrt{3}/2 \\ 1/2 \end{matrix}}_{\vec{u}} & \underbrace{\begin{matrix} -1/2 \\ \sqrt{3}/2 \end{matrix}}_{\vec{v}} \end{bmatrix}$.

Day 17

U is unitary & real.

$$\underbrace{B \begin{bmatrix} \vec{u} & \vec{v} \end{bmatrix}}_{U} = \begin{bmatrix} \underbrace{3\vec{u}}_{B\vec{u}} & \underbrace{4\vec{v}}_{B\vec{v}} \end{bmatrix} \underbrace{\left(= \underbrace{\begin{bmatrix} \vec{u} & \vec{v} \end{bmatrix}}_{U} \underbrace{\begin{bmatrix} 3 & 0 \\ 0 & 4 \end{bmatrix}}_{A} \right)}_{\text{optional information}}$$

$$\Rightarrow B \underbrace{\left(= UA\underbrace{U^{-1}}_{U^t} \right)}_{\text{optional information}} = \begin{bmatrix} \underbrace{\begin{matrix} 3\sqrt{3}/2 \\ 3/2 \end{matrix}}_{3\vec{u}} & \underbrace{\begin{matrix} -2 \\ 2\sqrt{3} \end{matrix}}_{4\vec{v}} \end{bmatrix} \underbrace{\begin{bmatrix} \sqrt{3}/2 & 1/2 \\ -1/2 & \sqrt{3}/2 \end{bmatrix}}_{U^{-1} = U^t}$$

$$\Rightarrow B = \begin{bmatrix} 9/4 + 1 & 3\sqrt{3}/4 - \sqrt{3} \\ 3\sqrt{3}/4 - \sqrt{3} & 3/4 + 3 \end{bmatrix}$$

Day 17

(2) $\dim(C(D)) = \#\text{pivot columns} = 4$

$\dim(R(D)) = \underbrace{\#\text{pivot rows}}_{= \#\text{pivots}} = 4$ too.

$\dim(N(D)) = \#\text{ non-pivot columns} = 3$

$\dim(L(D)) = \#\text{ non-pivot rows} = 2$

(3) True. If E is $n \times n$, then

$\dim(N(E)) = n - r = \dim(L(E))$

where $r = \#\text{pivots}$.

① Find $\det\left(\begin{bmatrix} 1 & 0 & 0 & 0 \\ 5 & 2 & 0 & 0 \\ 6 & 7 & 3 & 0 \\ 8 & 9 & 10 & 4 \end{bmatrix}\right)$ & $\det\left(\begin{bmatrix} -1 & 4 & -i & 9 \\ 0 & i & 7 & 0 \\ 0 & 0 & 2 & -6 \\ 0 & 0 & 0 & 3 \end{bmatrix}\right)$.

Day 19

These are easier than they might look.

② Suppose $B = \begin{bmatrix} 1 & 5 & 3 \\ 0 & 4 & 0 \\ 0 & 0 & 7 \end{bmatrix} = E_3 E_2 E_1 A$ where

E_1 performs row operation $-R_2 + R_3$,
E_2 performs row operation $R_1 \longleftrightarrow R_2$,
and E_3 performs $\frac{1}{2} R_1$. What is $\det(A)$?

③ Give examples of 2×2 matrices A & B such that $\det(A+B) \neq \det(A) + \det(B)$.

④ Suppose A is 5×5 and $\det(A) = 3$.
What is $\det(2A)$? $\det(A^2)$? $\det(A^{-1})$?

⑤ What is $\det\left(\begin{bmatrix} 1 & 2 \\ 5 & 7 \end{bmatrix}\right)$?

Day 19

(1) $1 \cdot 2 \cdot 3 \cdot 4 = \boxed{24}$ & $-1 \cdot i \cdot 2 \cdot 3 = \boxed{-6i}$

(2) $28 = 1 \cdot 4 \cdot 7 = \det(B) = \det(E_3)\det(E_2)\det(E_1)\det(A)$

$= (\mathbf{1/2})(-1)(1)\det(A)$

$\Rightarrow \boxed{-56} = \det(A)$

(3) $\det\left(\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}\right) = 1 \neq 0 + 0 = \det\left(\begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}\right) + \det\left(\begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}\right)$

$\parallel$

$\det\left(\begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix} + \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}\right)$

So, $A = \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}$ & $B = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}$ work.

(4) $\det(2A) = \det(E_5)\det(E_4)\det(E_3)\det(E_2)\det(E_1)\det(A)$

$= 2 \cdot 2 \cdot 2 \cdot 2 \cdot 2 \cdot 3 = \boxed{96}$

where E_k performs row operation $2R_k$.

Day 19

$\det(AA) = \det(A)\det(A) = 3 \cdot 3 = \boxed{9}$

$\det(A^{-1}) = \det(A)^{-1} = 3^{-1} = \boxed{1/3}$

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5 $\det\left(\begin{bmatrix} 1 & 2 \\ 5 & 7 \end{bmatrix}\right) = 1 \cdot 7 - 2 \cdot 5 = \boxed{-3}$

Day 20

① Find the eigenvalues of $A = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 2 & 0 & 0 \\ 0 & 0 & 3 & 0 \\ 1 & 4 & 9 & 4 \end{bmatrix}$.

For each eigenvalue, find an eigenvector.

② Find the eigenvalues of $\begin{bmatrix} 5 & 1 \\ 1 & -5 \end{bmatrix}$.

For each eigenvalue, find an eigenvector.

③ Repeat ② for $\begin{bmatrix} 5 & -1 \\ 1 & 5 \end{bmatrix}$.

④ Find a basis for the eigenspace $\begin{bmatrix} 4 & 1 & 0 \\ 0 & 4 & 0 \\ 0 & 0 & 4 \end{bmatrix}$

has for its unique eigenvalue.

⑤ Repeat ② for $\begin{bmatrix} 5 & 0 \\ 1 & 5 \end{bmatrix}$.

⑥ Give an example of a matrix with an eigenvalue that has algebraic multiplicity 4 & geometric multiplicity 2.

Day 20

① $\lambda = 1, 2, 3, 4$ are the eigenvalues.

$$A - I = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 2 & 0 \\ 1 & 4 & 9 & 3 \end{bmatrix} \xrightarrow{RREF} \begin{bmatrix} 1 & 0 & 0 & 3 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

$x_1 + 3x_4 = 0$
$x_2 = 0$
$x_3 = 0$

Choosing $x_4 = -1$ (for example), $\begin{bmatrix} 3 \\ 0 \\ 0 \\ -1 \end{bmatrix}$ is an eigenvector for $\lambda = 1$.

$$A - 2I = \begin{bmatrix} -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 1 & 4 & 9 & 2 \end{bmatrix} \xrightarrow{RREF} \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 1/2 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

$x_1 = 0$
$x_2 + \frac{1}{2} x_4 = 0$
$x_3 = 0$

Choosing $x_4 = -2$ (for example), $\begin{bmatrix} 0 \\ 1 \\ 0 \\ -2 \end{bmatrix}$ is an eigenvector for $\lambda = 2$.

$$A - 3I = \begin{bmatrix} -2 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 1 & 4 & 9 & 1 \end{bmatrix} \xrightarrow{RREF} \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & -1/9 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

Choosing $x_4 = -9$ (for example), $\begin{bmatrix} 0 \\ 0 \\ 1 \\ -9 \end{bmatrix}$ is an eigenvector for $\lambda = 3$.

$$A-4I = \begin{bmatrix} -3 & 0 & 0 & 0 \\ 0 & -2 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 1 & 4 & 9 & 0 \end{bmatrix} \xrightarrow{\text{RREF}} \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \quad \begin{array}{l} x_1 = 0 \\ x_2 = 0 \\ x_3 = 0 \end{array}$$

Day 20

Choosing (for example) $x_4 = 1$, $\begin{bmatrix} 0 \\ 0 \\ 0 \\ 1 \end{bmatrix}$ is an eigenvector for 4.

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(2) $\det\left(\begin{bmatrix} 5-\lambda & 1 \\ 1 & -5-\lambda \end{bmatrix}\right) = (5-\lambda)(-5-\lambda) - (1)(1) = \lambda^2 - 26$

The eigenvalues are $\pm\sqrt{26}$.

$$(\sqrt{26}-5)(-5-\sqrt{26}) + 1 = 0$$

$$\begin{bmatrix} 5-\sqrt{26} & 1 \\ 1 & -5-\sqrt{26} \end{bmatrix} \xrightarrow[(\sqrt{26}-5)R_2 + R_1]{} \begin{bmatrix} 0 & 0 \\ 1 & -5-\sqrt{26} \end{bmatrix} \quad x_1 - (5+\sqrt{26})x_2 = 0$$

$$\begin{bmatrix} 5+\sqrt{26} & 1 \\ 1 & -5+\sqrt{26} \end{bmatrix} \xrightarrow[-(5+\sqrt{26})R_2 + R_1]{} \begin{bmatrix} 0 & 0 \\ 1 & -5+\sqrt{26} \end{bmatrix} \quad x_1 + (\sqrt{26}-5)x_2 = 0$$

Choosing $x_2 = 1$: $\begin{bmatrix} 5+\sqrt{26} \\ 1 \end{bmatrix}$ is an eigenvector for $\sqrt{26}$.

& $\begin{bmatrix} 5-\sqrt{26} \\ 1 \end{bmatrix}$ is an eigenvector for $-\sqrt{26}$.

Day 20

(3) $\begin{bmatrix} 5-\lambda & -1 \\ 1 & 5-\lambda \end{bmatrix}$ has det. $(5-\lambda)^2+1$.

$0=(5-\lambda)^2+1 \iff \lambda - 5 = \pm\sqrt{-1} \iff \lambda = 5 \pm i$.

$\begin{bmatrix} 5-(5+i) & -1 \\ 1 & 5-(5+i) \end{bmatrix} = \begin{bmatrix} -i & -1 \\ 1 & -i \end{bmatrix} \xrightarrow[iR_2+R_1]{} \begin{bmatrix} 0 & 0 \\ 1 & -i \end{bmatrix}$

$x_1 - ix_2 = 0$

So, choosing $x_2 = 1$, $\begin{bmatrix} i \\ 1 \end{bmatrix}$ is an eigenvector for $5+i$.

Similar computations show that $\begin{bmatrix} -i \\ 1 \end{bmatrix}$ is an eigenvector

$5-i$, which is not surprising: $A\vec{v}=\lambda\vec{v} \Rightarrow A\bar{\vec{v}} = \bar{\lambda}\bar{\vec{v}}$

when A is real.

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(4) $\begin{bmatrix} 4-4 & 1 & 0 \\ 0 & 4-4 & 0 \\ 0 & 0 & 4-4 \end{bmatrix} = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$ has null space basis $\vec{e}_1, \vec{e}_3$.

$x_2 = 0$

So, $\vec{e}_1, \vec{e}_3$ is an <u>eigenbasis</u> for $\lambda = 4$ & $A = \begin{bmatrix} 4 & 1 & 0 \\ 0 & 4 & 0 \\ 0 & 0 & 4 \end{bmatrix}$.

Day 20

(5) $\det\left(\begin{bmatrix} 5-\lambda & 0 \\ 1 & 5-\lambda \end{bmatrix}\right) = (5-\lambda)^2 = 0 \iff \boxed{\lambda = 5}$

$N\left(\begin{bmatrix} 5-5 & 0 \\ 1 & 5-5 \end{bmatrix}\right) = \{\vec{x} \mid x_1 = 0\}$ includes eigenvector $\begin{bmatrix} 0 \\ 1 \end{bmatrix}$.

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(6) $\begin{bmatrix} 7 & 3 & 0 & 0 \\ 0 & 7 & 5 & 0 \\ 0 & 0 & 7 & 0 \\ 0 & 0 & 0 & 7 \end{bmatrix} = A$ has eigenvalue 7 which has

algebraic multiplicity 4 since $\det(A - \lambda I) = (7-\lambda)^4$

& geometric multiplicity 2 since $A - 7I = \begin{bmatrix} 0 & 3 & 0 & 0 \\ 0 & 0 & 5 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$

$\xrightarrow{\text{RREF}} \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$ has 2 non-pivot columns.

$\begin{matrix} F & P & P & F \end{matrix}$