Decide if each of these statements is true or false. Give a counterexample for each false statement.

Day 21

① If λ is an eigenvalue for A, then λ is an eigenvalue for A^t.

② If λ is an eigenvalue for A, then λ is an eigenvalue for A^*.

③ If λ is an eigenvalue for A, then $\overline{\lambda}$ is an eigenvalue for $\overline{A}$.

④ If $\vec{v}$ is an eigenvector for A, then $\overline{\vec{v}}$ is an eigenvector for $\overline{A}$.

⑤ If $\vec{v}$ is an eigenvector for A and A is real, then $\vec{v}$ is an eigenvector for A^t.

Day
21

⑥ If $\vec{v}$ is an eigenvector for A,
then $\vec{v}$ is an eigenvector for A^2.

⑦ If $\vec{v}$ is an eigenvector for eigenvalue λ
and matrix A, then $\lambda \neq 0$.

⑧ If $\vec{v}$ is an eigenvector for A and
A is invertible, then $\vec{v}$ is an eigenvector for A^{-1}.

⑨ If λ is an eigenvalue for A,
then λ is an eigenvalue for A^2.

⑩ If 0 is an eigenvalue of A,
then A is singular.

⑪ If A is singular (and square),
then 0 is an eigenvalue of A.

⑫ If $\vec{v}$ is an eigenvector for A & B both $n \times n$,
then $\vec{v}$ is an eigenvector for AB.

Day 21

⑬ If λ is an eigenvalue for A & B both $n \times n$, then λ^2 is an eigenvalue for AB.

⑭ If $\vec{v}$ is an eigenvector for A & B both $n \times n$, then $\vec{v}$ is an eigenvector for $7A + 4B$.

⑮ If λ is an eigenvalue of A & B both $n \times n$, then 2λ is an eigenvalue of $A + B$.

⑯ If A is real and 3×3, then A has a real eigenvalue.

⑰ If $A^2 = I$ and λ is an eigenvalue of A, then λ is 1 or -1.

⑱ If 1 is the only eigenvalue of A, then $A = I$.

⑲ If λ is an eigenvalue for A, ~~then~~ then $\lambda^2 + \lambda - 6$ is an eigenvalue for $(A + 3I)(A - 2I)$.

①–⑨: Suppose $\det(A-\lambda I)=0$, $\vec{v}\neq 0$, and $A\vec{v}=\lambda\vec{v}$.

Day 21

① True. $\det(A^t-\lambda I)=\det(A^t-\lambda I^t)=\det((A-\lambda I)^t)$
$=\det(A-\lambda I)=0$

② False. $\begin{bmatrix} i & 0 \\ 0 & i \end{bmatrix}$, but not $\begin{bmatrix} -i & 0 \\ 0 & -i \end{bmatrix}=\begin{bmatrix} i & 0 \\ 0 & i \end{bmatrix}^*$,
has eigenvalue i.

③ True. $\det(\bar{A}-\bar{\lambda} I)=\det(\bar{A}-\bar{\lambda}\bar{I})=\det(\overline{A-\lambda I})$
$=\overline{\det(A-\lambda I)}=\bar{0}=0$.

④ True. $\bar{A}\bar{\vec{v}}=\overline{A\vec{v}}=\overline{\lambda\vec{v}}=\bar{\lambda}\bar{\vec{v}}$

⑤ False. $\begin{bmatrix} 1 & 0 \\ 1 & 1 \end{bmatrix}$ has unique eigenspace $N\left(\begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix}\right)=\langle\vec{e}_2\rangle$,
in particular, $\vec{e}_2$ is an eigenvector. But $\begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}$
has unique eigenspace $N\left(\begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}\right)=\langle\vec{e}_1\rangle$, which excludes $\vec{e}_2$.

Day 21

⑥ True. $(AA)\vec{v} = A(\lambda\vec{v}) = \lambda(A\vec{v}) = (\lambda\lambda)\vec{v}$

⑦ False. $\begin{bmatrix}0&0\\0&0\end{bmatrix}\begin{bmatrix}1\\0\end{bmatrix} = 0\begin{bmatrix}1\\0\end{bmatrix}$

⑧ True. [illegible] $A\vec{v}=\lambda\vec{v} \Rightarrow \vec{v} = A^{-1}A\vec{v} = A^{-1}\lambda\vec{v} = \lambda A^{-1}\vec{v}$

$\Rightarrow \lambda^{-1}\vec{v} = A^{-1}\vec{v}$. ($\lambda \neq 0$ because $\det(A-0I)$
$\neq 0$ because A^{-1} exists.)

⑨ False. $\begin{bmatrix}3&0\\0&2\end{bmatrix}$ has eigenvalue 3; $\begin{bmatrix}9&0\\0&4\end{bmatrix}$ doesn't.

⑩ True. $0=\det(A-0I) \Rightarrow 0=\det(A) \Rightarrow A$ singular.

⑪ True. A singular $\Rightarrow \exists\, \vec{v} \in N(A)-\{\vec{0}\} = N(A-0I)-\{\vec{0}\}$

⑫ True. $AB\vec{v} = A\mu\vec{v} = \mu A\vec{v} = \mu\lambda\vec{v}$
if $A\vec{v}=\lambda\vec{v}$ & $B\vec{v}=\mu\vec{v}$.

⑬ False. $1=1^2$ is an eigenvalue of $\begin{bmatrix}1&1\\0&1\end{bmatrix}$ & $\begin{bmatrix}1&0\\1&1\end{bmatrix}$,
[illegible] but not for $\begin{bmatrix}1&1\\0&1\end{bmatrix}\begin{bmatrix}1&0\\1&1\end{bmatrix} = \begin{bmatrix}2&1\\1&1\end{bmatrix}$.

Day 21

(14) True. $(7A+4B)\vec{v}=(7\lambda+4\mu)\vec{v}$ if $A\vec{v}=\lambda\vec{v}$ & $B\vec{v}=\mu\vec{v}$.

(15) False. $\begin{bmatrix}1&0\\0&0\end{bmatrix}$ & $\begin{bmatrix}0&0\\0&1\end{bmatrix}$ have eigenvalue 1, but $\begin{bmatrix}1&0\\0&1\end{bmatrix}$ does not have eigenvalue 1+1.

(16) True. $\det(A-\lambda I)$ is a real polynomial $p(\lambda)$ $=-\lambda^3+\dots$ with odd degree. So, $\lim_{\lambda\to-\infty}p(\lambda)=+\infty$, $\lim_{\lambda\to\infty}p(\lambda)=-\infty$, and $p(\lambda)=0$ somewhere in between.

(17) True. $A\vec{v}=\lambda\vec{v}\Rightarrow A^2\vec{v}=\lambda^2\vec{v}\Rightarrow I\vec{v}=\lambda^2\vec{v}$
$\Rightarrow\vec{v}=\lambda^2\vec{v}\Rightarrow(\vec{v}=\vec{0}\text{ or }1=\lambda^2)$

(18) False. $\begin{bmatrix}1&1\\0&1\end{bmatrix}$.

(19) True. $(A+3I)(A-2I)\vec{v}=(A+3I)(\lambda-2)\vec{v}=\overbrace{(\lambda+3)(\lambda-2)}^{\lambda^2+\lambda-6}\vec{v}$

Day 22

① Which of these matrices are diagonalizable?

$$A=\begin{bmatrix}2&1&0\\0&2&0\\0&0&3\end{bmatrix},\quad B=\begin{bmatrix}2&0&1\\0&2&0\\0&0&3\end{bmatrix},$$

$$C=\begin{bmatrix}2&0&0\\0&3&1\\0&1&3\end{bmatrix},\quad D=\begin{bmatrix}1&4&5\\0&2&6\\0&0&3\end{bmatrix},\quad E=\begin{bmatrix}2&0&0\\0&2&1\\0&0&3\end{bmatrix}$$

---

② Given $A=\begin{bmatrix}2&9\\9&2\end{bmatrix}$, find $\sqrt{A}$, A^{100}, e^{A} and e^{tA}.

③ If $dx/dt = 2x+9y$ and $dy/dt = 9x+2y$ and $x(0)=0$ and $y(0)=-3$, find $x(t)$ & $y(t)$.

~~④ If A is diagonalizable, 5×5, $A^5=A$, $A^4 \neq I$, and A is real~~

Day 22

④ Give two examples of a real 4×4 matrix with eigenvalues $3i$ & $-3i$ and no others. Make the first diagonalizable and the second not.

---

⑤
$$1 + 1 + 1 = 3 = t_1$$
$$1 + 1 + 3 = 5 = t_2$$
$$1 + 3 + 5 = 9 = t_3$$
$$3 + 5 + 9 = 17 = t_4$$
$$5 + 9 + 17 = 31 = t_5$$

and so on...

$$t_n = \alpha_1 \lambda_1^n + \alpha_2 \lambda_2^n + \alpha_3 \lambda_3^n$$ for all $n = 1, 2, 3, \ldots$

if you get the six constants $\alpha_1, \alpha_2, \alpha_3, \lambda_1, \lambda_2, \lambda_3$ right. Find a polynomial $p(\lambda)$ whose roots are $\lambda_1, \lambda_2,$ and λ_3. ~~[illegible]~~

① Not A: $\det(A-\lambda I) = -(\lambda-2)^{\textcircled{2}}(\lambda-3)$ but

HW 22

$$\dim(N(A-2I)) = \dim\Big(N\underbrace{\left(\begin{bmatrix}0&1&0\\0&0&0\\0&0&1\end{bmatrix}\right)}_{\{\vec{x} \mid x_2=x_3=0\} = \langle\{\vec{e}_1\}\rangle}\Big) = 1 < 2.$$

Yes B: $\det(B-\lambda I) = -(\lambda-2)^{\textcircled{2}}(\lambda-3)$ and

$$\dim(N(B-2I)) = \dim\Big(N\underbrace{\left(\begin{bmatrix}0&0&1\\0&0&0\\0&0&1\end{bmatrix}\right)}_{\{\vec{x} \mid x_3=0\} = \langle\{\vec{e}_1, \vec{e}_2\}\rangle .}\Big) = 2.$$

Yes C:

$$\det(C-\lambda I) = (2-\lambda)((3-\lambda)^2-1)$$
$$= (2-\lambda)(2-\lambda)(4-\lambda)$$
$$= (2-\lambda)^{\textcircled{2}}(4-\lambda)$$

$$\dim(N(C-2I)) = \dim\Big(N\underbrace{\left(\begin{bmatrix}0&0&0\\0&1&1\\0&1&1\end{bmatrix}\right)}_{\{\vec{x} \mid x_2+x_3=0\} = \langle\{\vec{e}_1, \vec{e}_2-\vec{e}_3\}\rangle}\Big) = 2$$

HW 22

Yes D: $\det(D-\lambda I) = \underbrace{(1-\lambda)(2-\lambda)(3-\lambda)}_{\text{no repeated roots.}}$

Yes E: $\det(E-\lambda I) = (2-\lambda)((2-\lambda)(3-\lambda)-0\cdot 1)$

$= (2-\lambda)^{②}(3-\lambda)$ and

$\dim(N(E-2I)) = \dim\Big(\underbrace{N\Big(\begin{bmatrix}0&0&0\\0&0&1\\0&0&1\end{bmatrix}\Big)}_{\{\vec{x} \mid x_3=0\} = \langle\{\vec{e}_1,\vec{e}_2\}\rangle}\Big) = 2$

---

② $\det(A-\lambda I) = (2-\lambda)^2 - 9^2 = (2-\lambda+9)(2-\lambda-9)$

$= (11-\lambda)(-7-\lambda)$. $N(A-11I) = N\Big(\begin{bmatrix}-9&9\\9&-9\end{bmatrix}\Big)$

$= N\Big(\begin{bmatrix}1&-1\\0&0\end{bmatrix}\Big) = \Big\langle\Big\{\begin{bmatrix}1\\1\end{bmatrix}\Big\}\Big\rangle$. $N(A+7I) = N\Big(\begin{bmatrix}9&9\\9&9\end{bmatrix}\Big)$

$\xrightarrow{RREF}$

$= N\Big(\begin{bmatrix}1&1\\0&0\end{bmatrix}\Big) = \Big\langle\Big\{\begin{bmatrix}1\\-1\end{bmatrix}\Big\}\Big\rangle$. $A = \begin{bmatrix}1&1\\1&-1\end{bmatrix}\begin{bmatrix}11&0\\0&-7\end{bmatrix}\begin{bmatrix}1&1\\1&-1\end{bmatrix}^{-1}$.

$\begin{bmatrix}1&1\\1&-1\end{bmatrix}^{-1} = \dfrac{1}{(1)(-1)-(1)(1)}\begin{bmatrix}-1&-1\\-1&+1\end{bmatrix} = \begin{bmatrix}1/2&1/2\\1/2&-1/2\end{bmatrix}$

$\boxed{\text{HW } 22}$

$$\sqrt{A} = \begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix} \begin{bmatrix} \sqrt{11} & 0 \\ 0 & \sqrt{-7} \end{bmatrix} \begin{bmatrix} 1/2 & 1/2 \\ 1/2 & -1/2 \end{bmatrix}$$

$$= \frac{1}{2} \begin{bmatrix} \sqrt{11} + i\sqrt{7} & \sqrt{11} - i\sqrt{7} \\ \sqrt{11} - i\sqrt{7} & \sqrt{11} + i\sqrt{7} \end{bmatrix}$$

$$A^{100} = \begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix} \begin{bmatrix} 11^{100} & 0 \\ 0 & (-7)^{100} \end{bmatrix} \begin{bmatrix} 1/2 & 1/2 \\ 1/2 & -1/2 \end{bmatrix}$$

$$= \begin{bmatrix} 11^{100} + 7^{100} & 11^{100} - 7^{100} \\ 11^{100} - 7^{100} & 11^{100} + 7^{100} \end{bmatrix} \cdot \frac{1}{2}$$

$$e^{A} = \begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix} \begin{bmatrix} e^{11} & 0 \\ 0 & e^{-7} \end{bmatrix} \begin{bmatrix} 1/2 & 1/2 \\ 1/2 & -1/2 \end{bmatrix}$$

$$= \begin{bmatrix} (e^{11} + e^{-7})/2 & (e^{11} - e^{-7})/2 \\ (e^{11} - e^{-7})/2 & (e^{11} + e^{-7})/2 \end{bmatrix}$$

$$e^{tA} = \frac{1}{2} \begin{bmatrix} e^{11t} + e^{-7t} & e^{11t} - e^{-7t} \\ e^{11t} - e^{-7t} & e^{11t} + e^{-7t} \end{bmatrix}$$

HW 22

③ $$\frac{d}{dt}\begin{bmatrix} x \\ y \end{bmatrix} = A\begin{bmatrix} x \\ y \end{bmatrix} \quad \text{where} \quad A = \begin{bmatrix} 2 & 9 \\ 9 & 2 \end{bmatrix}$$

$$\begin{bmatrix} x(t) \\ y(t) \end{bmatrix} = e^{tA}\begin{bmatrix} x(0) \\ y(0) \end{bmatrix} = \frac{-3}{2}\begin{bmatrix} e^{11t} - e^{-7t} \\ e^{11t} + e^{-7t} \end{bmatrix}$$

---

④ $$\begin{bmatrix} 0 & -3 & 0 & 0 \\ 3 & 0 & 0 & 0 \\ 0 & 0 & 0 & -3 \\ 0 & 0 & 3 & 0 \end{bmatrix} \quad \& \quad \begin{bmatrix} 0 & -3 & 0 & 0 \\ 3 & 0 & 1 & 0 \\ 0 & 0 & 0 & -3 \\ 0 & 0 & 3 & 0 \end{bmatrix}$$

---

⑤ $$\begin{bmatrix} t_{n+3} \\ t_{n+2} \\ t_{n+1} \end{bmatrix} \begin{matrix} = \\ = \\ = \end{matrix} \begin{bmatrix} t_{n+2} + t_{n+1} + t_n \\ t_{n+2} \\ t_{n+1} \end{bmatrix} = \begin{bmatrix} 1 & 1 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix}\begin{bmatrix} t_{n+2} \\ t_{n+1} \\ t_n \end{bmatrix}$$

$$\det\left(\begin{bmatrix} 1-\lambda & 1 & 1 \\ 1 & -\lambda & 0 \\ 0 & 1 & -\lambda \end{bmatrix}\right) = (1-\lambda)\det\left(\begin{bmatrix} -\lambda & 0 \\ 1 & -\lambda \end{bmatrix}\right)$$

$$- 1 \det\left(\begin{bmatrix} 1 & 0 \\ 0 & -\lambda \end{bmatrix}\right)$$

$$+ 1 \det\left(\begin{bmatrix} 1 & -\lambda \\ 0 & 1 \end{bmatrix}\right)$$

$$= (1-\lambda)\lambda^2 + \lambda + 1$$

Day 26

① Let $F: P_3 \to P_3$ where

$$(Fp)(x) = (x^2-1)p''(x) + 7xp'(x+5) - p'(3) + p'''(8x^2) - (x^2p(x))'' + \int_{x^2}^{2} p'''(x+1)\,dx.$$

Is F linear?

② Let $G: M_{25} \to M_{52}$, $GA = A^t$.

Is G linear?

③ Let $H: M_{44} \to M_{44}$, $HA = A^* + A$.

Is H linear?

④ Let $J: M_{33} \to M_{33}$, $JA = 3A + I$.

Is J linear?

Day 26

⑤ Let $K: P_2 \rightarrow P_4$ where $(Kp)(x) = (p(x))^2$. Is K linear?

⑥ Find the matrix of $T: U \rightarrow L$ where $U = \{A \in M_{33} \mid A \text{ is upper triangular}\}$ & $L = \{A \in M_{33} \mid A \text{ is lower triangular}\}$ & $TA = A^t$, *after* finding bases of U & L. (The matrix of T depends on which bases you choose.)

⑦ Let $Q_1: M_{22} \rightarrow M_{22}$ where Q_1A is A after adding column 2 to column 1.
For basis $\begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}$, $\begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}$, $\begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix}$, $\begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}$,
Find the matrix of Q_1.

Day 26

⑧ Let $Q_2 : M_{22} \to M_{22}$ where $Q_2 A$ is A after adding row 2 to row 1. Using the basis from ⑦, find the matrix of Q_2.

⑨ Give an example $A \in M_{22}$ where $Q_1 Q_2 A \neq Q_2 Q_1 A$.

⑩ Define $Q_1 Q_2 - Q_2 Q_1 : M_{22} \to M_{22}$ by $(Q_1 Q_2 - Q_2 Q_1) A = Q_1 Q_2 A - Q_2 Q_1 A$. Find a basis for $\{A \in M_{22} \mid Q_1 Q_2 A = Q_2 Q_1 A\}$ by first finding a basis for the null space of the matrix of $Q_1 Q_2 - Q_2 Q_1$ with respect to the basis from ⑦.

Yes for ①, ②. No for ③, ④, ⑤.

Day 26

③ : $H(iI_4) = 0 \neq 2iI_4 = iH(I_4)$

④ : $J(0+0) = I \neq 2I = J0 + J0$

⑤ : $K(1+x) = 1+2x+x^2 \neq 1+x^2 = K1 + Kx$

⑥ Input basis:

$$\underbrace{\begin{bmatrix}1&0&0\\0&0&0\\0&0&0\end{bmatrix}}_{\vec{u}_1}, \underbrace{\begin{bmatrix}0&1&0\\0&0&0\\0&0&0\end{bmatrix}}_{\vec{u}_2}, \underbrace{\begin{bmatrix}0&0&1\\0&0&0\\0&0&0\end{bmatrix}}_{\vec{u}_3}, \underbrace{\begin{bmatrix}0&0&0\\0&1&0\\0&0&0\end{bmatrix}}_{\vec{u}_4}, \underbrace{\begin{bmatrix}0&0&0\\0&0&1\\0&0&0\end{bmatrix}}_{\vec{u}_5}, \underbrace{\begin{bmatrix}0&0&0\\0&0&0\\0&0&1\end{bmatrix}}_{\vec{u}_6}$$

output basis: $T\vec{u}_1, T\vec{u}_2, T\vec{u}_3, T\vec{u}_4, T\vec{u}_5, T\vec{u}_6$

$$\begin{bmatrix}1&0&0\\0&0&0\\0&0&0\end{bmatrix}, \begin{bmatrix}0&0&0\\1&0&0\\0&0&0\end{bmatrix}, \begin{bmatrix}0&0&0\\0&0&0\\1&0&0\end{bmatrix}, \begin{bmatrix}0&0&0\\0&1&0\\0&0&0\end{bmatrix}, \begin{bmatrix}0&0&0\\0&0&0\\0&1&0\end{bmatrix}, \begin{bmatrix}0&0&0\\0&0&0\\0&0&1\end{bmatrix}$$

Matrix: $I_6 = \begin{bmatrix}1&0&0&0&0&0\\0&1&0&0&0&0\\0&0&1&0&0&0\\0&0&0&1&0&0\\0&0&0&0&1&0\\0&0&0&0&0&1\end{bmatrix}$

⑦

Day 26

$Q_1 \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}$ has coordinates $\begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \end{bmatrix}$

$Q_1 \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} = \begin{bmatrix} 1 & 1 \\ 0 & 0 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} + \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}$ ~~has coordinates~~

$Q_1 \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix}$ has coordinates $\begin{bmatrix} 0 \\ 0 \\ 1 \\ 0 \end{bmatrix}$

$Q_1 \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 1 & 1 \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix} + \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}$ has coord's $\begin{bmatrix} 0 \\ 0 \\ 1 \\ 1 \end{bmatrix}$

$$(\text{Matrix of } Q_1) = \begin{bmatrix} 1 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

---

⑧

$Q_2 \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}$ has coordinates $\begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \end{bmatrix}$

$Q_2 \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}$ has coordinates $\begin{bmatrix} 0 \\ 1 \\ 0 \\ 0 \end{bmatrix}$

$Q_2 \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 1 & 0 \end{bmatrix}$ has coordinates $\begin{bmatrix} 1 \\ 0 \\ 1 \\ 0 \end{bmatrix}$

$Q_2 \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ 0 & 1 \end{bmatrix}$ has coordinates $\begin{bmatrix} 0 \\ 1 \\ 0 \\ 1 \end{bmatrix}$

$$\underbrace{\begin{bmatrix} 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}}_{\text{Matrix of } Q_2}$$

Day
26

⑨ & ⑩: Sorry, I meant for $Q_1Q_2 \neq Q_2Q_1$! But actually

$$Q_1Q_2\begin{bmatrix}a & b\\ c & d\end{bmatrix} = \begin{bmatrix}a+b+c+d & b+d\\ c+d & d\end{bmatrix} = Q_2Q_1\begin{bmatrix}a & b\\ c & d\end{bmatrix}.$$

So, $\{A \in M_{22} \mid Q_1Q_2A = Q_2Q_1A\} = M_{22}$ has the same basis as given in ⑦, for example.

The matrix of $Q_1Q_2 - Q_2Q_1$ is just $\begin{bmatrix}0&0&0&0\\0&0&0&0\\0&0&0&0\\0&0&0&0\end{bmatrix}$.

Day
27

① • Give an example of a linear transformation $T_1 : \mathbb{C}^4 \to M_{22}$ that is invertible.

• Then a $T_2 : \mathbb{C}^4 \to M_{22}$ that's not injective.

• Is there $T_3 : \mathbb{C}^4 \to M_{22}$ that is surjective but not injective? If "yes," give an example. If not, explain why.

② Suppose we have linear transformations as listed below. Fill in the blanks with "yes," "no," or "maybe."

	injective?	surjective?	invertible?
$T_4 : P_5 \to M_{32}$			
$T_5 : M_{44} \to M_{35}$			
$T_6 : \mathbb{C}^9 \to \mathbb{C}^{10}$			

Day 27

③ Is $T_7 : M_{22} \to M_{22}$ where
$T_7 A = 5A + 7A^t$ injective? surjective?

④ Is $T_8 : M_{22} \to \mathbb{C}^3$ where

$$T \begin{bmatrix} a & b \\ c & d \end{bmatrix} = \begin{bmatrix} a+b+c \\ b+c+d \\ c+d+a \end{bmatrix}$$

injective? surjective?

⑤ Is $T_9 : P_2 \to P_4$ where

$$(Tp)(x) = x^2 p(x+1) - xp'(1)$$

injective? surjective?

① $T_1\begin{bmatrix}a\\b\\c\\d\end{bmatrix}=\begin{bmatrix}a&b\\c&d\end{bmatrix}$. $T_2\begin{bmatrix}a\\b\\c\\d\end{bmatrix}=\begin{bmatrix}a&b\\c&0\end{bmatrix}$.

Day 27

There is no surjective non-injective linear $T_3: \mathbb{C}^4 \to M_{22}$. If there were, T_3 would have $\underbrace{\text{four}}_{\dim(M_{22})}$ pivot rows but fewer than $\underbrace{\text{four}}_{\dim(\mathbb{C}^4)}$ pivot columns in the RREF of its matrix (with respect to any input basis & output basis).

But #(pivot rows) = # pivots = #(pivot columns).

②

$T: V \to W$	$\dim(V)$	$\dim(W)$	injective?	surjective?	invertible?
$T_4: P_5 \to M_{32}$	6	6	maybe	maybe	maybe
$T_5: M_{44} \to M_{35}$	16	15	no	maybe	no
$T_6: \mathbb{C}^9 \to \mathbb{C}^{10}$	9	10	maybe	no	no

Day 27

③ For basis $\begin{bmatrix}1&0\\0&0\end{bmatrix}, \begin{bmatrix}0&1\\0&0\end{bmatrix}, \begin{bmatrix}0&0\\1&0\end{bmatrix}, \begin{bmatrix}0&0\\0&1\end{bmatrix}$,

$T_7\begin{bmatrix}a&b\\c&d\end{bmatrix} = \begin{bmatrix}5a+7a & 5b+7c\\ 5c+7b & 5d+7d\end{bmatrix}$ has matrix

$\begin{bmatrix}12&0&0&0\\0&5&7&0\\0&7&5&0\\0&0&0&12\end{bmatrix}$ which has det. $(12)[5\cdot5\cdot12-7\cdot7\cdot12]$

$\neq 0$ and, therefore, an RREF of $I_4 = \begin{bmatrix}1&0&0&0\\0&1&0&0\\0&0&1&0\\0&0&0&1\end{bmatrix}$.

Hence, T_7 is injective & surjective.

---

④ For input basis as in ③ & output basis $\vec{e}_1, \vec{e}_2, \vec{e}_3$,

T_8 has matrix $\begin{bmatrix}1&1&1&0\\0&1&1&1\\1&0&1&1\end{bmatrix} \xrightarrow[\substack{-R_1+R_3\\ \text{then}\\ -R_2+R_1\\ R_2+R_3\\ \text{then}\\ -R_3+R_2}]{}$

$$\to \begin{array}{ccccc} P & P & P & & \\ \hline 1 & 0 & 0 & -1 & P \\ 0 & 1 & 0 & -1 & P \\ 0 & 0 & 1 & 2 & P \end{array}.$$

$\boxed{\text{So, } T_8 \text{ is surjective but not injective.}}$

⑤ $T_9(ax^2+bx+c) = ax^2(x^2+2x+1) + bx^2(x+1)$
$+ cx^2 - x(2a+b) = ax^4 + (2a+b)x^3$
$+ (a+b+c)x^2 + (-2a-b)x + 0\cdot 1$

Day
27

For input basis $x^2, x, 1$ & output basis $x^4, x^3, x^2, x, 1$,

$$(\text{matrix of } T_9) = \begin{bmatrix} 1 & 0 & 0 \\ 2 & 1 & 0 \\ 1 & 1 & 1 \\ -2 & -1 & 0 \\ 0 & 0 & 0 \end{bmatrix} \xrightarrow{RREF} \begin{array}{c} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \begin{array}{c} P \\ P \\ P \\ \\ \\ \end{array} \\ \begin{array}{ccc} P & P & P \end{array} \end{array}$$

T_9 is injective but not surjective.

Day 28

① Let $T : M_{22} \to M_{22}$ where $TA = A^t$. Give a basis of M_{22} that makes the matrix of T diagonal.

② $R : \mathbb{C}^3 \to \mathbb{C}^3$ with matrix $\begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{bmatrix}$ with respect to basis $\vec{e}_1, \vec{e}_2, \vec{e}_3$ is actually a 120° rotation. Find an orthonormal basis of $\mathbb{C}^3$ with respect to which the matrix of R is

$$\begin{bmatrix} \cos\frac{2\pi}{3} & -\sin\frac{2\pi}{3} & 0 \\ \sin\frac{2\pi}{3} & \cos\frac{2\pi}{3} & 0 \\ 0 & 0 & 1 \end{bmatrix}.$$

③ Explain why there is no basis of P_3 that makes the matrix of $D : P_3 \to P_3$, where $(Dp)(x) = p'(x)$, diagonal.

Day 28

④ Suppose $F: P_2 \rightarrow P_3$ has matrix

$$A = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 2 \\ 1 & 1 & 0 \\ 0 & 0 & -1 \end{bmatrix}$$

with respect to [input basis $1, x+1, x^2+x+1$] & [output basis $(x-1)^3, (x-1)^2, x-1, 5$]. Find the matrix B of F with respect to input basis $1, x, x^2$ & output basis $1, x, x^2, x^3$.

⑤ Which of the following matrices have orthonormal bases of eigenvectors?

$$\begin{bmatrix} 2 & 1 \\ 1 & 5 \end{bmatrix}, \begin{bmatrix} 2 & -1 \\ 1 & 5 \end{bmatrix}, \begin{bmatrix} 5 & -1 \\ 1 & 5 \end{bmatrix}, \begin{bmatrix} 0 & 0 \\ 0 & i \end{bmatrix},$$

$$\begin{bmatrix} 0 & -i \\ i & 1 \end{bmatrix}, \begin{bmatrix} 0 & i \\ i & 1 \end{bmatrix}$$

Day 28

① For basis $\begin{bmatrix}1&0\\0&0\end{bmatrix}, \begin{bmatrix}0&1\\0&0\end{bmatrix}, \begin{bmatrix}0&0\\1&0\end{bmatrix}, \begin{bmatrix}0&0\\0&1\end{bmatrix}$, T has matrix $\begin{bmatrix}1&0&0&0\\0&0&1&0\\0&1&0&0\\0&0&0&1\end{bmatrix}$ which has eigenvalue -1 with eigenbasis $\vec{e}_2-\vec{e}_3$ and eigenvalue $+1$ with ~~eigenbasis~~ eigenbasis $\vec{e}_1, \vec{e}_4, \vec{e}_2+\vec{e}_3$. Therefore, for basis $\begin{bmatrix}0&1\\-1&0\end{bmatrix}, \begin{bmatrix}1&0\\0&0\end{bmatrix}, \begin{bmatrix}0&0\\0&1\end{bmatrix}, \begin{bmatrix}0&1\\1&0\end{bmatrix}$, T has diagonal matrix $\begin{bmatrix}-1&0&0&0\\0&1&0&0\\0&0&1&0\\0&0&0&1\end{bmatrix}$.

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② For our desired basis $\vec{u}_1, \vec{u}_2, \vec{u}_3$, we have $R\vec{u}_3=\vec{u}_3$. So, $\vec{u}_3 \in N(R-I)=N\left(\begin{bmatrix}-1&1&0\\0&-1&1\\1&0&-1\end{bmatrix}\right)$

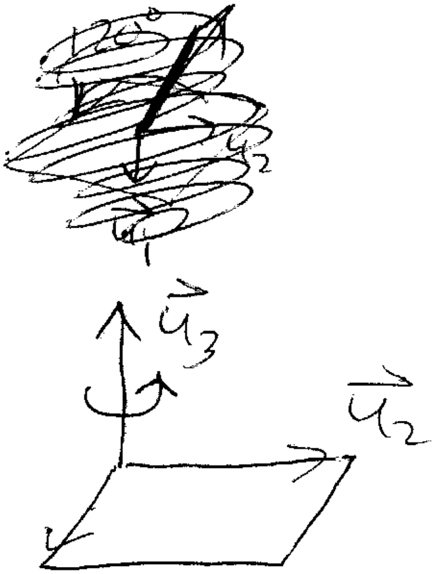
$$= N\left(\begin{bmatrix} 1 & 0 & -1 \\ 0 & 1 & -1 \\ 0 & 0 & 0 \end{bmatrix}\right) = \left\langle \left\{ \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} \right\} \right\rangle.$$ Let $\vec{v}_3 = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$.

Day 28

$$x_1 - x_3 = 0$$
$$x_2 - x_3 = 0$$

Let $\vec{u}_3 = \frac{1}{\sqrt{3}} \vec{v}_3$.

So, R rotates 120° around the axis parallel to $\vec{v}_3$ through the origin, since the axis is the only thing a 120° rotation doesn't change. Any two other vectors $\vec{u}_1, \vec{u}_2$ such that $\vec{u}_1, \vec{u}_2, \vec{u}_3$ is orthonormal should be rotated 120°.



Apply Gram-Schmidt and then normalization to $\vec{v}_3, \vec{e}_1, \vec{e}_2$ (any basis including $\vec{v}_3$ will work): $\vec{w}_1 = \vec{v}_3$, $\vec{w}_2 = \vec{e}_1 - \frac{\langle \vec{w}_1, \vec{e}_1 \rangle}{\langle \vec{w}_1, \vec{w}_1 \rangle} \vec{w}_1$,

$$\vec{w}_3 = \vec{e}_2 - \frac{\langle \vec{w}_1, \vec{e}_2 \rangle}{\langle \vec{w}_1, \vec{w}_1 \rangle} \vec{w}_1 - \frac{\langle \vec{w}_2, \vec{e}_2 \rangle}{\langle \vec{w}_2, \vec{w}_2 \rangle} \vec{w}_2,$$

$$\vec{u}_3 = \|\vec{w}_1\|^{-1} \vec{w}_1, \quad \vec{u}_1 = \|\vec{w}_2\|^{-1} \vec{w}_2, \quad \vec{u}_2 = \|\vec{w}_3\|^{-1} \vec{w}_3.$$

Day 28

②

$$\vec{w}_1 = \begin{bmatrix}1\\1\\1\end{bmatrix}, \quad \vec{w}_2 = \begin{bmatrix}1\\0\\0\end{bmatrix} - \frac{1}{3}\begin{bmatrix}1\\1\\1\end{bmatrix} = \begin{bmatrix}2/3\\-1/3\\-1/3\end{bmatrix},$$

$$\vec{w}_3 = \begin{bmatrix}0\\1\\0\end{bmatrix} - \frac{1}{3}\begin{bmatrix}1\\1\\1\end{bmatrix} - \frac{-1/3}{2/3}\begin{bmatrix}2/3\\-1/3\\-1/3\end{bmatrix} = \begin{bmatrix}0\\1/2\\-1/2\end{bmatrix},$$

$$\vec{u}_3 = \frac{1}{\sqrt{3}}\begin{bmatrix}1\\1\\1\end{bmatrix}, \quad \vec{u}_1 = \begin{bmatrix}\sqrt{2/3}\\-1/\sqrt{6}\\-1/\sqrt{6}\end{bmatrix}, \quad \vec{u}_2 = \begin{bmatrix}0\\1/\sqrt{2}\\-1/\sqrt{2}\end{bmatrix}$$

$$R\vec{u}_1 = \begin{bmatrix}-1/\sqrt{6}\\-1/\sqrt{6}\\\sqrt{2/3}\end{bmatrix} = -\frac{1}{2}\vec{u}_1 - \frac{\sqrt{3}}{2}\vec{u}_2$$

But we want $R\vec{u}_1 = \left(\cos\frac{2\pi}{3}\right)\vec{u}_1 + \left(\sin\frac{2\pi}{3}\right)\vec{u}_2 = -\frac{1}{2}\vec{u}_1 + \frac{\sqrt{3}}{2}\vec{u}_2.$

Reordering $\vec{u}_1$ & $\vec{u}_2$ fixes this:

$$\left\{\begin{aligned} R\vec{u}_2 &= \begin{bmatrix}1/\sqrt{2}\\-1/\sqrt{2}\\0\end{bmatrix} = -\frac{1}{2}\vec{u}_2 + \frac{\sqrt{3}}{2}\vec{u}_1 = \left(\cos\frac{2\pi}{3}\right)\vec{u}_2 + \left(\sin\frac{2\pi}{3}\right)\vec{u}_1 \\ R\vec{u}_1 &= \begin{bmatrix}-1/\sqrt{6}\\-1/\sqrt{6}\\\sqrt{2/3}\end{bmatrix} = -\frac{\sqrt{3}}{2}\vec{u}_2 - \frac{1}{2}\vec{u}_1 = \left(-\sin\frac{2\pi}{3}\right)\vec{u}_2 + \left(\cos\frac{2\pi}{3}\right)\vec{u}_1 \end{aligned}\right.$$

So, $\vec{u}_2, \vec{u}_1, \vec{u}_3 = \begin{bmatrix}0\\1/\sqrt{2}\\-1/\sqrt{2}\end{bmatrix}, \begin{bmatrix}\sqrt{2}/\sqrt{3}\\-1/\sqrt{6}\\-1/\sqrt{6}\end{bmatrix}, \begin{bmatrix}1/\sqrt{3}\\1/\sqrt{3}\\1/\sqrt{3}\end{bmatrix}$ works.

For basis $x^3, x^2, x, 1$, D has matrix

$A = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 3 & 0 & 0 & 0 \\ 0 & 2 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{bmatrix}$ which has unique eigenvalue 0 with algebraic multiplicity 4 from $\det(A - \lambda I) = \lambda^{\textcircled{4}}$ and geometric multiplicity $\dim(N(A - 0I)) = 1$ because

$$A - 0I = A \xrightarrow{\text{RREF}} \begin{array}{cccc} \left[\begin{matrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{matrix}\right] \\ \begin{matrix} P & P & P & F \end{matrix} \end{array}. \qquad 1 < 4.$$

So, D cannot be diagonalized. And no other basis would fix this: if D has a diagonal matrix

$\begin{bmatrix} \lambda_1 & 0 & 0 & 0 \\ 0 & \lambda_2 & 0 & 0 \\ 0 & 0 & \lambda_3 & 0 \\ 0 & 0 & 0 & \lambda_4 \end{bmatrix}$ for some basis $\vec{v}_1, \vec{v}_2, \vec{v}_3, \vec{v}_4$, then

$[Dp = \lambda p \;\&\; p \neq 0]$ would still have solutions only when $\lambda = 0$, implying $\lambda_1 = \lambda_2 = \lambda_3 = \lambda_4 = 0$ and $D = 0$, which it is not.

$$\begin{bmatrix} \quad B \quad \end{bmatrix} = \begin{bmatrix} \quad C \quad \end{bmatrix} \begin{bmatrix} \quad A \quad \end{bmatrix} \begin{bmatrix} \quad D \quad \end{bmatrix}$$

Day 28

④

$$\begin{array}{c} \text{output} \\ 1 \\ x \\ x^2 \\ x^3 \\ \text{basis} \end{array} \xleftarrow{F} \begin{array}{c} \text{input} \\ 1 \\ x \\ x^2 \\ \text{basis} \end{array} \qquad \begin{array}{c} \text{output} \\ 1 \\ x \\ x^2 \\ x^3 \\ \text{basis} \end{array} \xleftarrow{id} \begin{array}{c} \text{input} \\ (x-1)^3 \\ (x-1)^2 \\ x-1 \\ 5 \\ \text{basis} \end{array} \qquad \begin{array}{c} \text{output} \\ (x-1)^3 \\ (x-1)^2 \\ (x-1) \\ 5 \\ \text{basis} \end{array} \xleftarrow{F} \begin{array}{c} \text{input} \\ 1 \\ x+1 \\ x^2+x+1 \\ \text{basis} \end{array} \qquad \begin{array}{c} \text{output} \\ 1 \\ x+1 \\ x^2+x+1 \\ \text{basis} \end{array} \xleftarrow{id} \begin{array}{c} \text{input} \\ 1 \\ x \\ x^2 \\ \text{basis} \end{array}$$

$$C = \begin{bmatrix} -1 & 1 & -1 & 5 \\ 3 & -2 & 1 & 0 \\ -3 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{bmatrix}$$

$$(x-1)^2 = 1 - 2x + x^2$$

$$(x-1)^3 = -1 + 3x - 3x^2 + x^3$$

$$\begin{bmatrix} 1 & 1 & 1 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} \quad D^{-1} \quad \end{bmatrix}$$

$$\begin{array}{ccc} \uparrow & \uparrow & \uparrow \\ 1 & 1+x & 1+x+x^2 \end{array}$$

$$\begin{array}{c} 1 \\ x \\ x^2 \\ \text{output} \\ \text{basis} \end{array} \xleftarrow{id} \begin{array}{c} 1 \\ x+1 \\ x^2+x+1 \\ \text{input} \\ \text{basis} \end{array}$$

$$\left[D^{-1} \mid I_3 \right] \xrightarrow[\text{RREF}]{} \left[I_3 \mid D \right]$$

$$\left[\underbrace{\begin{matrix}1 & 1 & 1\\ 0 & 1 & 1\\ 0 & 0 & 1\end{matrix}}_{D^{-1}}\ \underbrace{\begin{matrix}1 & 0 & 0\\ 0 & 1 & 0\\ 0 & 0 & 1\end{matrix}}_{I_3}\right] \xrightarrow{-R_2+R_1} \begin{bmatrix}1 & 0 & 0 & 1 & -1 & 0\\ 0 & 1 & 1 & 0 & 1 & 0\\ 0 & 0 & 1 & 0 & 0 & 1\end{bmatrix} \xrightarrow{-R_3+R_2} \left[\underbrace{\begin{matrix}1 & 0 & 0\\ 0 & 1 & 0\\ 0 & 0 & 1\end{matrix}}_{I_3}\ \underbrace{\begin{matrix}1 & -1 & 0\\ 0 & 1 & -1\\ 0 & 0 & 1\end{matrix}}_{D}\right]$$

④ Day 28

$$B = CAD = \begin{bmatrix}-1 & 1 & -1 & 5\\ 3 & -2 & 1 & 0\\ -3 & 1 & 0 & 0\\ 1 & 0 & 0 & 0\end{bmatrix} \underbrace{\begin{bmatrix}1 & 0 & 0\\ 0 & 0 & 2\\ 1 & 1 & 0\\ 0 & 0 & -1\end{bmatrix}\begin{bmatrix}1 & -1 & 0\\ 0 & 1 & -1\\ 0 & 0 & 1\end{bmatrix}}$$

$$= \begin{bmatrix} & \cdots & \end{bmatrix}\begin{bmatrix}1 & -1 & 0\\ 0 & 0 & 2\\ 1 & 0 & -1\\ 0 & 0 & -1\end{bmatrix} = \boxed{\begin{bmatrix}\overset{-2}{\cancel{0}} & +1 & -2\\ 4 & -3 & -5\\ -3 & 3 & 2\\ 1 & -1 & 0\end{bmatrix}}$$

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⑤ Test $AA^* \overset{?}{=} A^*A$.

$\begin{bmatrix}2 & 1\\ 1 & 5\end{bmatrix}$, ~~$\begin{bmatrix}2 & -1\\ 1 & 5\end{bmatrix}$~~, $\begin{bmatrix}5 & -1\\ 1 & 5\end{bmatrix}$,

$\begin{bmatrix}0 & 0\\ 0 & i\end{bmatrix}$, $\begin{bmatrix}0 & -i\\ i & 1\end{bmatrix}$ pass.

Note: $A = A^* \Rightarrow AA^* = A^*A$.

~~Also note: $A^*A = (AA^*)^*$ so $AA^* = A^*A \Leftrightarrow AA^* = (AA^*)^*$.~~

Day 28

⑤

$$\begin{bmatrix} 2 & 1 \\ 1 & 5 \end{bmatrix}^* = \begin{bmatrix} 2 & 1 \\ 1 & 5 \end{bmatrix}.$$

$$\left.\begin{aligned} \begin{bmatrix} 2 & -1 \\ 1 & 5 \end{bmatrix}\begin{bmatrix} 2 & 1 \\ -1 & 5 \end{bmatrix} &= \begin{bmatrix} 5 & -3 \\ -3 & 26 \end{bmatrix}. \\ \begin{bmatrix} 2 & 1 \\ -1 & 5 \end{bmatrix}\begin{bmatrix} 2 & -1 \\ 1 & 5 \end{bmatrix} &= \begin{bmatrix} 5 & 3 \\ 3 & 26 \end{bmatrix} \end{aligned}\right\} \text{fail}$$

$$\begin{bmatrix} 5 & -1 \\ 1 & 5 \end{bmatrix}\begin{bmatrix} 5 & 1 \\ -1 & 5 \end{bmatrix} = \begin{bmatrix} 26 & 0 \\ 0 & 26 \end{bmatrix} = \begin{bmatrix} 5 & 1 \\ -1 & 5 \end{bmatrix}\begin{bmatrix} 5 & -1 \\ 1 & 5 \end{bmatrix}$$

$$\begin{bmatrix} 0 & 0 \\ 0 & i \end{bmatrix}\begin{bmatrix} 0 & 0 \\ 0 & -i \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & -i \end{bmatrix}\begin{bmatrix} 0 & 0 \\ 0 & i \end{bmatrix}$$

$$\begin{bmatrix} 0 & -i \\ i & 1 \end{bmatrix}^* = \begin{bmatrix} 0 & -i \\ i & 1 \end{bmatrix}$$

$$\left.\begin{aligned} \begin{bmatrix} 0 & i \\ i & 1 \end{bmatrix}\begin{bmatrix} 0 & -i \\ -i & 1 \end{bmatrix} &= \begin{bmatrix} 1 & i \\ -i & 2 \end{bmatrix} \\ \begin{bmatrix} 0 & -i \\ -i & 1 \end{bmatrix}\begin{bmatrix} 0 & i \\ i & 1 \end{bmatrix} &= \begin{bmatrix} 1 & -i \\ i & 2 \end{bmatrix} \end{aligned}\right\} \text{fail}$$