8/27/12 Suggested exercises 1, 3, 5, 7 (p. 21)

Quiz on p. 1-8 on Wednesday 8/29

$\mathbb{Z} = \{\ldots, -3, -2, -1, 0, 1, 2, 3, \ldots\}$

$(\mathbb{Z}, +)$

$(\mathbb{Z}, \square)$ $\quad m \square n = m + n - 6$

$m \square n = (m-3) + (n-3)$

$f: \mathbb{Z} \to \mathbb{Z}, \quad f(x) = x - 3$

f is a bijection

$f(x+y) = x + y - 3 \neq \overbrace{f(x) \square f(y)}^{x+y-9}$

$g: \mathbb{Z} \to \mathbb{Z}, \quad g(x) = x + 3$

$g(x+y) = x + y + 3 \overset{?}{=} \cancel{(x+3)+(y+3)}$

$g(x) \square g(y)$

$(x+3)+(y+3)-6$

no.

$$g_Z(x+y) = x+y+z = g_z(x) \square g_z(y)$$

$$(x+z) \square (y+z)$$

$$x+z+y+z-6$$

$$\cancel{x}+\cancel{y}+z = \cancel{x}+\cancel{y}+2z-6$$

$$+6 \qquad\qquad +6$$

$$-z \qquad\qquad -z$$

$$6 = z$$

$h: \mathbb{Z} \to \mathbb{Z} \quad h(x) = x+6$

h is a bijection and

$$h(x+y) = h(x) \square h(y)$$

Check:

$$x+y+6 \overset{?}{=} (x+6) \square (y+6)$$

$$x+y+6 = (x+6)+(y+\cancel{6})-\cancel{6}$$

Yes.

We say h is an isomorphism from $(\mathbb{Z},+)$ to $(\mathbb{Z},\square)$.

Let's prove $x \square y = y \square x$,

using h:

$$
\begin{aligned}
x \square y &= h(h^{-1}(x)) \square h(h^{-1}(y)) \\
&= h^{-1}(x) + h^{-1}(y) \\
&= h^{-1}(y) + h^{-1}(x) \\
&= h(h^{-1}(y)) \square h(h^{-1}(x)) \\
&= y \square x
\end{aligned}
$$

In this sense, $(\mathbb{Z}, +)$ & $(\mathbb{Z}, \square)$ are the same: whatever we know about one transfers to the other.

For every two complete ordered fields $(\mathbb{R}_1, +_1, \cdot_1, <_1)$ & $(\mathbb{R}_2, +_2, \cdot_2, <_2)$, there is a unique bijection $f: \mathbb{R}_1 \to \mathbb{R}_2$ such that

$$f(x +_1 y) = f(x) +_2 f(y),$$

$$f(x \cdot_1 y) = f(x) \cdot_2 f(y),$$

and

$x <_1 y$ if and only if $f(x) <_2 f(y)$.

We call such an f an isomorphism from $(\mathbb{R}_1, +_1, \cdot_1, <_1)$ to $(\mathbb{R}_2, +_2, \cdot_2, <_2)$.