Prove: $\inf A = -\sup(-A)$ (#5 p. 21)

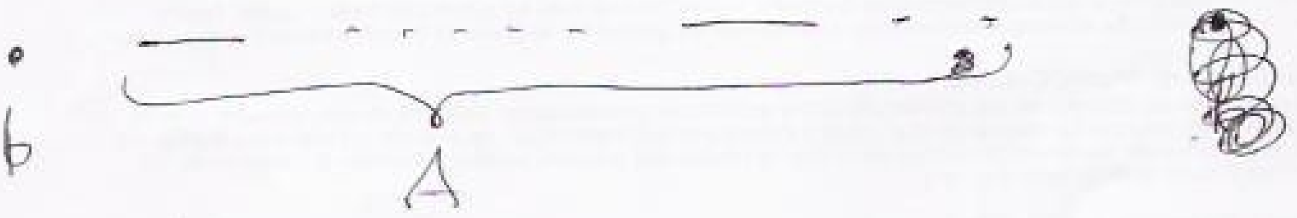
assumptions: $A \subset \mathbb{R}$, A bounded below,

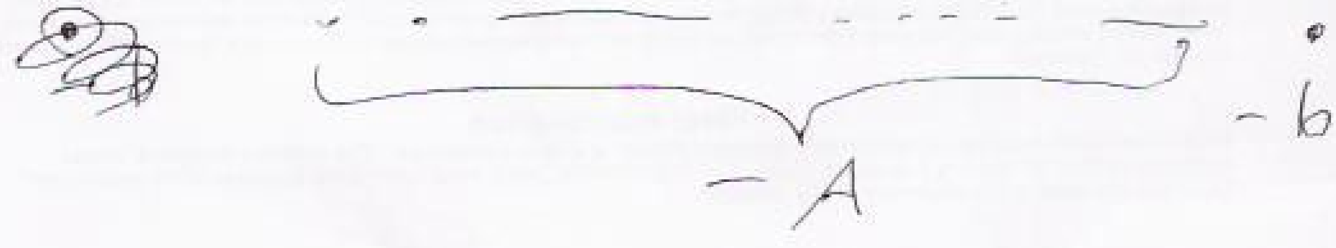
and A nonempty ($A \neq \emptyset$).

$-A = \{-x : x \in A\}$

We know $\mathbb{R}$ has the l.u.b. property (1.19):

$\forall E \subset \mathbb{R}$ if $\emptyset \neq E$ & E is bdd above, then $\sup E$ exists.





Let b be a lower bound of A, i.e., $\forall x \in A \quad b \leq x$. We claim $-b$ is an upper bound of $-A$. To see this, observe that if $y \in -A$, then $y = -x$ for some $x \in A$; hence, $y = -x \leq -b$. By the l.u.b. property of $\mathbb{R}$, there exists $s = \sup(-A)$. Now we need to show that $\inf(A)$ exists and equals $-s$...

Prove $1 = \sup\left\{\frac{n-1}{n} : n = 1, 2, 3, \dots\right\}$:

- 1 is an upper bound of

$$A = \left\{\frac{n-1}{n} : n = 1, 2, 3, \dots\right\}:$$

If $n = 1, 2, 3, \dots$, then

$$1 = \frac{n}{n} \;\&\; n \ge n-1, \text{ so } 1 \ge \frac{n-1}{n}. \checkmark$$

- 1 is the least upper bound of A, i.e., if $b \ge \frac{n-1}{n}$ for all $n = 1, 2, 3, \dots$, then $b \ge 1$:

~~We will show that $b - 1 \ge 0$, if $b \ge (n-1)/n$ for all $n = 1, 2, 3, \dots$~~

$$\cancel{b - 1 \ge \frac{n-1}{n} - 1 = \frac{n}{n} - \frac{1}{n} - \frac{n}{n} = -\frac{1}{n}.}$$

~~So, $b - 1 \ge -\frac{1}{n}$. So, $\frac{1}{b-1} \le -n$.~~

~~So, $\frac{1}{1-b} \ge n$, if $1 - b \ne 0$.~~

~~If $1 - b = 0$, then $b - 1 = 0 \ge 0$.~~

(1.20a) If $x \in \mathbb{R}$, $y \in \mathbb{R}$, and $x > 0$, then $\exists n \in \{1, 2, 3, \dots\}$ $nx > y$.

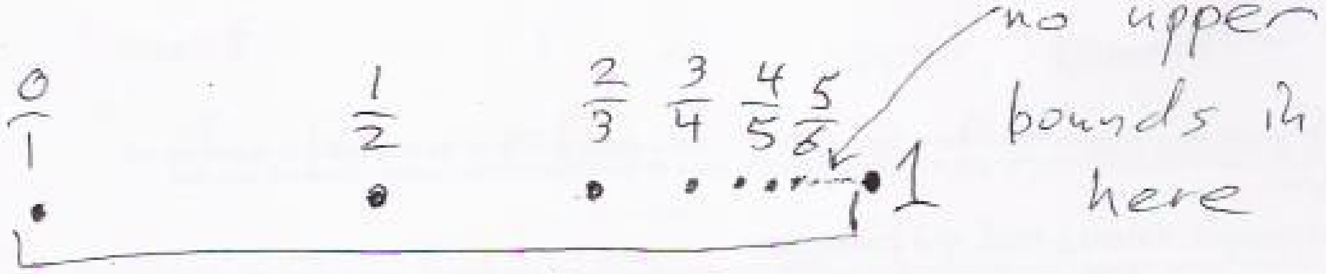
Seeking a contradiction, suppose $b < 1$ & $b \geq \frac{n-1}{n}$ for all $n = 1, 2, 3, \dots$

Then $0 < 1 - b$, so, for some $n \in \{1, 2, 3, \dots\}$, $n(1-b) > 1$, by 1.20a of Rudin with $x = 1 - b$ & $y = 1$.

So, $1 - b > \frac{1}{n}$; so, $1 > \frac{1}{n} + b$; so, $1 - \frac{1}{n} > b$; so, $\frac{n-1}{n} > b$, contradicting our assumption that $b \geq \frac{n-1}{n}$ for <u>all</u> $n = 1, 2, 3, \dots$

Thus 1 is the least upper bound. $\square$

Intuitive picture:



$$A = \left\{ \frac{n-1}{n} : n = 1, 2, 3, \dots \right\}$$

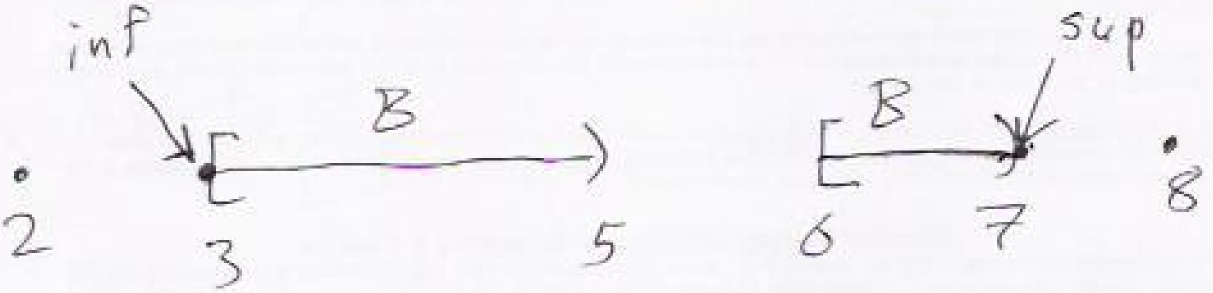
$$[3, 5) \cup [6, 7) = B$$

inf $B = 3 \in B$

sup $B = 7 \notin B$

2 is a lower bound of B

8 is an upper bound of B



Prove it: ~~Sup~~ If $x \in B$, then $3 \le x < 5$ or $6 \le x < 7$, so ~~In ei~~ $x < 5 < 7$ or $x < 7$, so 7 is an upper bound.

If c is an upper bound of B, then $\forall x \in [6, 7) \subset B$, we have $x \in B$, $\boxed{\text{so } x < c}$. So, if $c < 7$, then either $c \in [6, 7)$, in which case $c < c$ (impossible), or $c \in (-\infty, 6)$, in which case $c < 6 < c$, which is again impossible. Thus, $c < 7$ is impossible. So, 7 is the least upper bound.

insert: [To see this, observe that if $x \in [6, 7)$, then

$$6 \leq x < \frac{x+7}{2} < 7,$$

so $\frac{x+7}{2} \in B$, so $x < \frac{x+7}{2} \leq c$, so $x < c$.]

- Statement S says

  $\forall$ foo with glagnons,

  ($\forall$ = "for all" or "for every")

  $\exists$ bar that clinks that foo.

  ($\exists$ = "there exists")

- To refute S, find a foo with glagnons & prove that no bar clinks it.

- $S = \forall x$ (x is a foo w/ glagnons $\Rightarrow \exists y$ y bar & y clinks x)

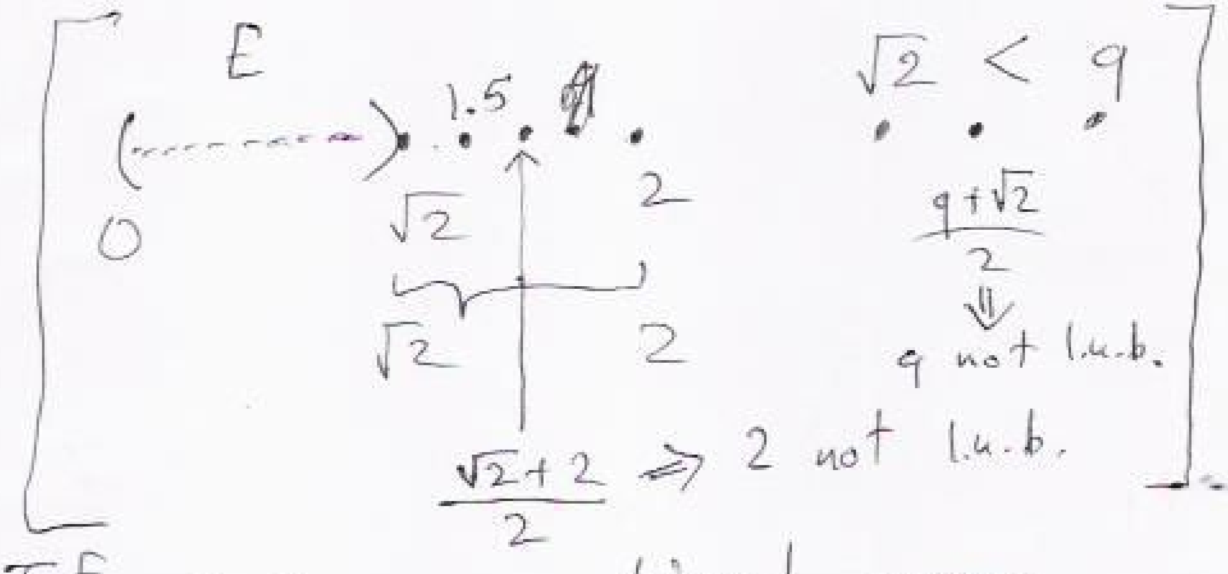
- $\neg S = \exists x$ (x is a foo w/ glagnons & $\forall y$ if y is a bar, then it doesn't clink x)

To prove $\mathbb{Q}$ lacks the l.u.b. property, find $E \subset \mathbb{Q}$ such that $E \neq \emptyset$ & E has an upper $\underbrace{\text{bound}}_{\text{in } \mathbb{Q}}$, but E has $\underbrace{\text{no l.u.b.}}_{\text{in } \mathbb{Q}}$

Let $E = (0, \sqrt{2}) \cap \mathbb{Q}$.

2 is an upper bound of E.

$1 \in E$, so $E \neq \emptyset$. Idea:



If q is a rational upper bound of $(0, \sqrt{2})$, then

$q > \sqrt{2}$. To see this, first note that $\sqrt{2} \notin \mathbb{Q}$, so $q \neq \sqrt{2}$.

If $q < \sqrt{2}$, then, by 1:20b, $\exists r \in \mathbb{Q}$ $q < r < \sqrt{2}$.

Let $s = \max(1, r) \geq r$, so that $s \in E \,(= (0, \sqrt{2}) \cap \mathbb{Q})$ & $q < r \leq s \in E$. Thus,

$q < \sqrt{2} \Rightarrow q$ not upper bound of E.

So, $q \not< \sqrt{2}$ & $q \neq \sqrt{2}$; so $\sqrt{2} < q$.

Therefore, $\frac{q + \sqrt{2}}{2} < q$ witnesses that q is not the least upper bound of E, as

$x < \sqrt{2} < (q + \sqrt{2})/2$ for all $x \in E$. $\square$

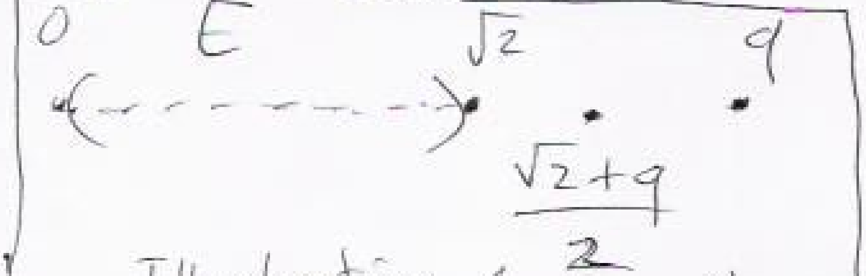


Illustration (optional)